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Quantum mechanical non-locality: the GHZ example

Consider the Hilbert space of three distinguishable spin-1/2 particles, neglecting spatial degrees of freedom. Because the particles are independent, the corresponding spin operators $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$ mutually commute: $[\mathbf{S}_1, \mathbf{S}_2] = [\mathbf{S}_2, \mathbf{S}_3] = [\mathbf{S}_3, \mathbf{S}_1] = 0$. For each particle, the spin operators satisfy the usual angular momentum commutation relation, e.g. $[S_1^x, S_1^y] = i\hbar S_1^z$. The Hilbert space is the tensor product of the three spin spaces, which is spanned by the eight products of the $S_{1,2,3}^z$ eigenstates, labeled in an obvious notation as

$$\Big\{|+++\rangle, |++-\rangle, |+-+\rangle, |-++\rangle, |--+\rangle, |-+-\rangle, |+--\rangle, |---\rangle\Big\}.$$

Consider now the three operators

$$A_1 = S_1^x S_2^y S_3^y, \qquad A_2 = S_1^y S_2^x S_3^y, \qquad A_3 = S_1^y S_2^y S_3^x. \tag{1}$$

For notational convenience, let us choose units with $\hbar/2 = 1$, so the eigenvalues of the spin operators are numerically ± 1.

1. Show that $[A_1, A_2] = [A_2, A_3] = [A_3, A_1] = 0$.

2. Show that $A_1^2 = A_2^2 = A_3^2 = 1$. Use this to show that the eigenvalues of $A_{1,2,3}$ are ± 1. Argue that specifying these three eigenvalues uniquely specifies the state of the three spin system.

3. From part 2 you know that there is a unique state with eigenvalue $+1$ for all three operators A_1, A_2, and A_3 (you will find this state below). Let us call this state $|Q\rangle$. Show that

$$B := S_1^x S_2^x S_3^x = -A_1 A_2 A_3, \tag{2}$$

so that $B|Q\rangle = -|Q\rangle$.

This system provides an example of a quantum mechanical violation of "Einstein locality", as explained nicely by N. David Mermin (Amer. J. Phys. **58** (1990) 731; Physics Today, June 1990, p. 9). (The example derives from work of D. Greenberger, M. Horne, and A. Zeilinger (GHZ).) Unlike the EPR experiments in which quantum mechanics violates locality on the level of *probabilities*, the GHZ example violates locality in a *single measurement.* The reasoning goes as follows. The result of a measurement of either S^x or S^y on any one of the particles is determined with certainty by the results of appropriately chosen S^x or S^y measurements on the other two particles, since the state is an eigenstate of the operators (1). Since the other two measurements can be distant and unable to causally influence the third measurement, the Einstein-Podolsky-Rosen (EPR) reality criterion[1] implies that there must exist "elements of reality" $m_1^x, m_1^y, m_2^x, m_2^y, m_3^x, m_2^y$, each having the value ± 1, which are revealed by a measurement of the corresponding spin operator. The product $m_1^x m_2^x m_3^x$ must be unity because it is equal to the product $(m_1^x m_2^y m_3^y)(m_1^y m_2^x m_3^y)(m_1^y m_2^y m_3^x)$, and each factor in parentheses is unity since the state is an eigenstate of the three operators (1) with eigenvalue $+1$. This implies that the result of measuring the operator B (2) is $+1$, in contradiction with quantum mechanics which asserts that the result is -1. Thus quantum mechanics is inconsistent with "local realism". That is, according to quantum mechanics there are properties of the physical world that cannot be reduced to properties of individual, local, subsystems.

4. Find $|Q\rangle$ in terms of the S^z eigenstates. One way to find this state is as follows:

- **(a)** Show that $A_1 A_2 = S_1^z S_2^z$, so that $S_1^z S_2^z |Q\rangle = |Q\rangle$, and similarly for 2,3 and 3,1.
- **(b)** Use part 4a to show that $|Q\rangle = a|+++\rangle + b|---\rangle$.
- **(c)** Show that $a = -b = 1/\sqrt{2}$ by requiring $B|Q\rangle = -|Q\rangle$.

[1] "If, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity." A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47** (1935) 777.