Addition of angular momenta:

Rotations in space are implemented on QM systems by unitary transformations
U(R)=exp(-i theta.J/hbar), where J^i are the hermitian generators of rotation.

J^i are also the angular momentum operators, and are conserved
if the Hamiltonian is invariant under rotations.

Rotation group structure implies [J^i,J^j] = ihbar epsilon^ijk J^k.

representations: can simultaneously diagonalize J_z and J^2, since [J_z,J^2]=0.
We analyzed this last semester.
Call the eigenstates |jm>,
where J_z |jm> = m |jm>, J^2|jm> = j(j+1) |jm>, with hbar=1 from now on.
The possible values of j are
0, 1/2, 1, 3/2, 2, ... and the possible values of m, for a given j, are j, j-1, j-2, ..., -j.
The representation with a given j is called the **"spin-j" representation**,
and it is 2j+1 dimensional.
These representations are **irreducible**, in the sense that there is no subspace
that is invariant (mapped into itself) under all rotations.
We can see this from the fact that
J_+ |jm> = Sqrt[j(j+1) - m(m+1)] |j,m+1> and J_- |jm> = Sqrt[j(j+1) - m(m-1)] |j,m-1>,
where J_+ = J_x + i J_y,   J_- = J_x - i J_y,
from which it is clear that by acting with rotations we move through all the states.

#example: 3d vectors V^i form the spin-1 rep. The tensor product of two of these
is the rank two tensors like V^i W^j, or more generally, T^ij. These are not irreducible.
Rather the antisymmetric part is by itself irreducible, and three dimensional,
hence another spin-1 rep. The symmetric part is reducible into the part proportional to
the Kronecker delta (trace) and the rest (symmetric trace-free part).
The trace part is the j=0 rep, the symm tracefree part is j=2
(since then 2j+1=5=number of independent components of a symmetric tracefree tensor).

example: 1/2 x 1/2 = 1 + 0, example: 1 x 1 = 2 + 1 + 0 (this is equivalent to the example above).

Note three different examples of spin-1 rep:
vector, antisymmetric tensor, |2p, m=-1,0,1> states of H-atom.
I.e., the rep is the abstract structure. Many things can realize it.

general scheme: j1xj2 spanned by basis {|j1m1>|j2m2>}. Decomposes into irreducibles.
Find by starting with top J_z state and working down with lowering operator J_-.
When fill out a rep, go back and start with the next highest top J_z state, which is the other
linear combination of the two second to two top J_z states.
This results in
j1 x j2 = (j_1+j_2) + (j_1+j_2 - 1) + ... + |j_1 -j_2|.

The largest spin rep, j1+j2, starts with top state equal to the product of the two top states |j1j1>|j2j2>.
To see that the smallest spin rep is |j1-j2|, suppose first that j1>=j2.

The argument I gave in class, cleaned up a bit here, was that the largest degeneracy
that occurs for fixed total m is   2j2+1, so there must be 2j2+1 different irreps in the decomposition.
Working our way down from the j1+j2 rep the last one must therefore be the j1-j2 rep.
A (sort of) different argument goes as follows. Each state |j1m1> must occur in every
rep, since acting with J_+ and J_- will eventually introduce it. In particular, |j1j1> must occur.
The smallest total m the state |j1j1>|j2m2> can have is if m2 is as small as possible, m2=-j2.
In this case, the total m is j1-j2, hence the smallest rep we have is spin- (j1-j2).
If j2>j1 then reverse the roles, and the smallest rep is spin-(j2-j1).
In general, we have that the smallest is spin-|j1-j2|. You can check that the total dimension
(2j1+1)(2j2+1) is equal to the sum over integer steps from j = |j1-j2| to j1+j2 of (2j+1).

|jm> = |m1m2><m1m2|jm> , sum on m1,m2 with m=m1+m2.
Similarly, |m1m2>= |jm><jm|m1m2>, where the sum is over j with m=m1+m2 fixed.
The expansion coefficients are the **Clebsch-Gordan coefficients.** The construction above
shows that they can always be taken to be real, so <m1m2|jm>=<jm|m1m2>*=<jm|m1m2>.
There is still an overall sign ambiguity of the CG coeffs, that is typically fixed by requiring that
the coefficient of |m1=j1>|m2=j-j1> in the the expansion of the top state |jj> of the spin-j rep. is positive,
i.e. <j1,j-j1|jj> is positive. (There is a typo in Baym in the fourth line after (15-40), where it reads
m1=j instead of m1=j1.)

Baym works out the case of   j x 1/2. There is a typo in eqn (15-44), which should have m2 = -/+1/2.)

The CG coeffs can be computed by:
-brute force
-Mathematica: ClebschGordan[{j1,m1},{j2,m2},{j,m}]   (**Note**: I mis-spelled it "Gordon" in class.)
-tables
-recursion relations
-a projection operator method
-amazingly enough, a CLOSED FORM formula has been found by Wigner for all the CG coeffs,
which was given in a more symmetrical form by Racah. See (106.14) of Landau & Lifshitz.
It is so complicated as to be unusable.