HW#8—Phys623—Spring 2017

Prof. Ted Jacobson
PSC 3151, (301)405-6020
jacobson@physics.umd.edu

1. *The 21 cm line*:

(a) Which type of multipole interaction dominates the $F = 1$ to $F = 0$ decay of the hyperfine of the ground state of hydrogen (a.k.a. the 21 cm line)? Why?

(b) Compute the decay rate for each of the three starting m_F values. (You may use a general formula, but should evaluate the relevant matrix element.) What is the lifetime in years? (The rate is the same for all three m_F. This is a consequence of spherical symmetry, but doesn't quite seem obvious to me, because the states with different m values are not related by rotations.)

(c) Suppose the initial state has $m_F = 1$. Calculate the polarization of the emitted photon if its wavevector is in the direction (i) $\hat{z}$, (ii) $-\hat{z}$, (iii) $\hat{x}$.

2. This problem concerns spontaneous dipole transitions of a hydrogen atom.

(a) Make an argument using dimensional analysis and basic physics to determine how the total rate Γ for an electric dipole transition of hydrogen depends upon e, c, $\hbar$, the frequency ω of the emitted photon, and the dipole matrix element $\mathbf{d} = \langle f|\mathbf{r}|i\rangle$. **Explain your logic**.

(b) Use your result from part 2a to make an order of magnitude estimate of the rate for the transition $2p \to 1s$. Next, compare this with the rate for the transitions $2p_{3/2} \to 2s_{1/2}$ and $2s_{1/2} \to 2p_{1/2}$. (The $2p_{3/2}$—$2s_{1/2}$ fine structure energy splitting is of order α^2 Rydbergs, and the $2s_{1/2}$—$2p_{1/2}$ Lamb shift is about ten times smaller than that.)

Possibly useful information:
$\alpha = e^2/\hbar c \simeq 1/137$
$\hbar \simeq 2/3$ eV-fs (1 fs = 10^{-15} s)
1 Rydberg = 13.6 eV

3. This problem concerns spontaneous decay of an excited state of hydrogen with the emission of one photon with momentum $\hbar\mathbf{k}$ and polarization vector ϵ with $\mathbf{k}\cdot\epsilon = 0$. The probability amplitude for such a transition between atomic states $|i\rangle$ and $|f\rangle$ is proportional to the dimensionless matrix element $\langle f|(M_{orb} + M_{spin})|i\rangle$ with

$$\langle f|M_{orb}|i\rangle = \frac{1}{mc}\,\epsilon^* \cdot \langle f|\mathbf{p}\,e^{-i\mathbf{k}\cdot\mathbf{r}}|i\rangle \quad (1)$$

$$\langle f|M_{spin}|i\rangle = \frac{i}{mc}\,(\mathbf{k}\times\epsilon^*) \cdot \langle f|\mathbf{S}\,e^{-i\mathbf{k}\cdot\mathbf{r}}|i\rangle \quad (2)$$

where $\mathbf{r}$ is the position vector of the electron, $\mathbf{p}$ is the momentum, and $\mathbf{S}$ is the spin.

(a) Show that, for the $2p \to 1s$ transition, $\langle 1s|(M_{orb} + M_{spin})|2p\rangle$ is of order α, where $\alpha = e^2/\hbar c \approx 1/137$ is the fine structure constant. (This leads to a lifetime of order 10^{-9} s for the $2p$ state.)

The $2s$ state has a much longer lifetime, of order $1/7$ s, because many types of transitions are forbidden. The remaining parts of this problem explore this phenomenon.

(b) Show that $\langle 1s|M_{orb}|2s\rangle = 0$.

(c) Consider the expansion of the spin transition matrix element $\langle f|M_{spin}|i\rangle = \sum_{n=0}^{\infty}(1/n!)\langle f|M^{(n)}_{spin}|i\rangle$, with $\langle f|M^{(n)}_{spin}|i\rangle = \frac{i}{mc}\left(\mathbf{k}\times\epsilon^*\right)\cdot\langle f|\mathbf{S}\left(-i\mathbf{k}\cdot\mathbf{r}\right)^n|i\rangle$. Show that $\langle 1s|M^{(n)}_{spin}|2s\rangle$ is zero for $n = 0, 1$ but non-zero for $n = 2$.

(d) Estimate the order of magnitude of $\langle 1s|M^{(2)}_{spin}|2s\rangle$ in powers of α.

(e) Using parts 3a and 3d estimate the lifetime of the 2s state assuming the transition proceeds via the matrix element $\langle 1s|M^{(2)}_{spin}|2s\rangle$.

(f) The actual lifetime (1/7 s) is much shorter than the lifetime you should have obtained in part 3e. Can you think of a reason? (It is *not* that the atom first decays via the allowed $2s_{1/2} \to 2p_{1/2}$ dipole transition. Why?)