Prof. Ted Jacobson
PSC 3151, (301)405-6020
jacobson@physics.umd.edu

1. *Scattering with non-free final states* A particle of mass m moves in a one-dimensional attractive potential $U(x) = -\gamma\delta(x)$, where $\delta(x)$ is the Dirac delta-function, and $\gamma > 0$. Use periodic boundary conditions with $x = \pm L$ identified, with $L \to \infty$. Since you're taking L to infinity, you need only retain the leading order contributions in $1/L$.[1]

(a) Find the wave function and the energy E_0 of the bound state. What is the parity of the wave function with respect to the operation $x \to -x$?

(b) Find the wave functions and the energies of the unbound states. Choose the wave functions to be symmetric or antisymmetric with respect to the parity operation $x \to -x$. [*Hint*: The symmetric states look like free states with a kink at $x = 0$, so can be written in the form $\psi_n(x) = \frac{1}{\sqrt{L}}\cos(k_n|x| + \phi_n)$.]

At time $t < 0$, the particle is in the ground state of the potential. At time $t > 0$, a small oscillating potential

$$V(t) = Ax^2 \sin(\omega t) \tag{1}$$

of frequency $\omega > |E_0|/\hbar$ is turned on.

(c) Which matrix elements of the perturbation (1) between the ground state and the symmetric or antisymmetric unbound states vanish because of the parity selection rule? Calculate the nonvanishing matrix elements.

(d) Using Fermi's golden rule, calculate the probability of transition of the particle from the ground state to an unbound state per unit time.

(e) Discuss and interpret the behavior of the ejection rate as ω approaches ∞, and as $\hbar\omega$ approaches $|E_0|$. Are there any other special values of ω? If so, why?

(f) How would the result change if for the final states you used free particle states rather than the exact unbound states? How does the difference behave in the limit $\omega \to \infty$? Interpret your result. [*Note*: When TJ solved this previously, he found a result that seems so strange it must be wrong: although the kink phase angle ϕ vanishes as $k \to \infty$, the ratio of the free particle matrix element to the exact state matrix element goes to $3/4$, so the kink in the exact state strongly affects the ejection rate no matter how high the energy is.]

[1] Note that we can choose units with $\hbar = m = \lambda = 0$. That simplifies the equations, but sacrifices the ability to catch errors using dimensional analysis. You may choose whether or not to live dangerously. If you do, you should double check yourself along the way, and restore the dimensionful constants at the end. I like the following method for restoring the constants. First, note that γ has dimensions of energy $\times$ length, so $l_u = \hbar^2/m\gamma$ has dimensions of length, $\omega_u = \gamma/\hbar l_u = m\gamma^2/\hbar^3$ has dimensions of frequency, and $A_u = \gamma/l_u^3 = m^3\gamma^4/\hbar^6$ has the dimensions of A, i.e. energy/length^2. Your result for the rate will be some function of the remaining constants, $\Gamma(\omega, A)$. To restore the other constants—i.e. to write the result in a form that holds in any system of units—you simply replace $\Gamma(\omega, A)$ by $\omega_u\Gamma(\omega/\omega_u, A/A_u)$. This is correct because it has the correct dimension (inverse time), and agrees with your result when using the adopted units.

2. *Canonical commutation relations for the electromagnetic field*: Check that if the vector potential operator is defined as in Schwabl (16.49), then the vector potential and the electric field are canonically conjugate at equal times, in the sense that

$$[A_i(\vec{x},t), E_j(\vec{y},t)] = -i(4\pi c)\hbar\delta^{\mathrm{T}}_{ij}\delta^3(\vec{x}-\vec{y}). \tag{2}$$

Here $\delta^{\mathrm{T}}_{ij} = \delta_{ij} - \partial_i\partial_j/\nabla^2$, where the derivatives act on the argument of the delta function on the right hand side of (2), yielding the "transverse delta function". (δ^{T}_{ij} is the projector that takes any vector field V_i to its transverse part, $V_i^T = \delta^{\mathrm{T}}_{ij}V_j$ (sum over j), so vanishes when contracted with the gradient of a scalar, $\delta^{\mathrm{T}}_{ij}\partial_j\lambda = 0$.)

3. *Quantum fluctuations of the electromagnetic field*

(a) Calculate the vacuum expectation value ("vev") of the electric field operator, $\langle 0|E_i(\vec{x},t)|0\rangle$.

(b) Calculate the vev of the product of two electric field operators at the same time but different positions, $\langle 0|E_i(\vec{x},t)E_j(\vec{x}',t)|0\rangle$. Convert the sum over wavevectors to an integral over $\vec{k}$ (in the infinite V limit), and change variables to extract the dependence on the separation distance $|\vec{x}-\vec{x}'|$.[2] Are the fluctuations at two spacelike related points correlated? What happens as the two points approach each other?

(c) Show that the Hamiltonian (16.48) is equivalent to (16.51a).

(d) Taking the vev of the Hamiltonian, find an expression for the energy density of the electromagnetic field in vacuum as an integral over k. If you cut off the upper limit of that integral at some k_c, what is the resulting energy density?

4. *Selection rules for atomic transitions*: Consider the matrix elements of the form

$$\langle\gamma' J' M_J' L' S'|\hat{T}|\gamma J M_J L S\rangle$$

where the states describe an atom with angular momentum quantum numbers $JLSM_J$, and remaining labels γ needed to specify a state, and the operator $\hat{T}$ is either the electric dipole, orbital magnetic dipole, spin magnetic dipole, or electric quadrupole transition operator. For the purposes of this problem, you only need to know that these operators are c-numbers times $\sum x^i$, $\sum L^i$, $\sum S^i$ and $\sum x^i x^j - \frac{1}{3}x^2\delta^{ij}$, respectively, where i and j are vector indices, the sums are over the electrons in the atom, and the letters have the usual meanings. The first three of these operators are irreducible tensor operators of rank 1, and the last is of rank 2. They are all tensor operators wrt $\vec{J}$, $\vec{L}$, and $\vec{S}$ (but the spin operator is scalar wrt $\vec{L}$ and the orbital operators are scalars wrt $\vec{S}$). Determine in each of these cases the joint selection rules for parity, J, M_J, and L. Make a table displaying the selection rules, and explain very briefly your reasoning.

[2] The polarization sum for a given k_i gives $\delta_{ij} - k_ik_j/k^2$. The remaining integral is proportional to $\delta_{ij} - \hat{n}_i\hat{n}_j$, where $\hat{n}$ is the unit vector in the direction of $\vec{x}-\vec{x}'$. You need not determine its numerical value after the variable change that extracts $|\vec{x}-\vec{x}'|$; but if you want to, you should insert a regulator $e^{-\epsilon k}$, evaluate the integral, and then take the limit $\epsilon \to 0$.