HW#6—Phys623—Spring 2017

Prof. Ted Jacobson
PSC 3151, (301)405-6020
jacobson@physics.umd.edu

1. Consider the Hamiltonian $H(t) = H_0 + \alpha(t)V$, where (generically) $[H_0, V] \neq 0$, and $\alpha(t)$ is a real function that vanishes if $|t| > \tau$.

(a) In the limit $\tau \to 0$, with $\gamma \equiv \int dt\, \alpha(t)$ fixed, the effect of V on the evolution is "impulsive". Find the state $|\psi(0_+)\rangle$ after the impulse in terms of the state $|\psi(0_-)\rangle$ before the impulse. (*Hint*: Make sure the evolution is unitary. It might help to consider first the case where $[H_0, V] = 0$.)

(b) Write the first order approximation for the state $|\psi(t)\rangle$ in terms of $|\psi(-\tau)\rangle$ using time-dependent perturbation theory.

(c) Take the impulsive limit of the first order perturbative result. Under what conditions does it give a good approximation to the exact impulsive result?

2. Consider the Hamiltonian given by $H(t) = H_0 + e^{\eta t}V_0$ for $t < 0$, and by $H(t) = H_0 + V_0$ for $t \geq 0$. Assume H_0 has discrete spectrum, and suppose the initial state at $t \to -\infty$ is an eigenstate $|n\rangle$ of H_0, with $H_0|n\rangle = E_n|n\rangle$.

(a) Show that in the limit $\eta \to 0$, the first order time-dependent shift of the state, evaluated at $t = 0$, agrees with the first order shift of the state in time-independent perturbation theory (with V_0 as the perturbation). (In fact, the exact time dependent solution coincides with an exact eigenstate of $H_0 + V_0$ in this limit. This is an instance of the *adiabatic theorem.*)

(b) How small must η be to ensure that the first order time-dependent shift of the state at $t = 0$ agrees to within 1% with the first order shift of the state in time-independent perturbation theory? It depends on the component $|m\rangle$ of the state, so give an answer for each m.

3. The time evolution of an harmonic oscillator acted on by any time dependent external force $F(t)$ can be found exactly in quantum mechanics, just as it can in classical mechanics. In this problem you'll learn how that works, and apply it to find the state at time t, if the system starts in the ground state at $t = 0$.

The force $F(t)$ is derived from the potential $-F(t)x = -f(t)(a + a^\dagger)$, where a and $a^\dagger$ are the ladder operators for the harmonic oscillator, and $f(t) = F(t)x_0/\sqrt{2}$. Here $x_0 = \sqrt{\hbar/m\omega}$, where m and ω are the mass and frequency of the oscillator. The Hamiltonian is thus

$$H = \hbar\omega(a^\dagger a + \tfrac{1}{2}) - f(t)(a + a^\dagger). \tag{1}$$

At this stage let's set $\hbar = 1$ by a units choice.

(a) Verify that, for a quantum system with Hamiltonian $H(t)$, the solution to the Schrodinger equation is given by $|\psi(t)\rangle = U(t, t')|\psi(t')\rangle$, where the time evolution operator $U(t, t')$ is defined to satisfy $i\partial_t U(t, t') = H(t)U(t, t')$, with the boundary condition $U(t, t) = I$.

(b) In the Heisenberg picture, the lowering operator evolves as $a_H(t) = U^\dagger a U$, where $U \equiv U(t,0)$, and $a = a_H(0)$ is the Schrodinger picture lowering operator. Show that a_H satisfies the differential equation $\dot{a}_H = -i\omega a_H + if$, and show that the solution is given by

$$a_H(t) = e^{-i\omega t}a + \alpha(t), \quad \alpha(t) = ie^{-i\omega t}\int_0^t dt' e^{i\omega t'} f(t'). \tag{2}$$

This is basically the complete solution to the dynamics, since any observable in the Heisenberg picture can be constructed from $a_H(t)$.

(c) Using the previous result, show that if the oscillator starts out in the ground state $|0\rangle$ at $t = 0$, the state thereafter satisfies $a|\psi(t)\rangle = \alpha(t)|\psi(t)\rangle$. That is, the ground state evolves to the "coherent state" $|\alpha(t)\rangle$.

(d) In case you haven't yet come across coherent states, or have forgotten how they work, show that the normalized coherent state defined by the condition $a|\alpha\rangle = \alpha|\alpha\rangle$ is given by

$$|\alpha\rangle := e^{-|\alpha|^2/2}\sum_n \frac{1}{\sqrt{n!}}\alpha^n|n\rangle, \tag{3}$$

where $|n\rangle$ is the number eigenstate.

(e) Find the probability $P_n(t)$ for the oscillator to be in its nth level at time t.

(f) Find $P_n(t)$ using first order time dependent perturbation theory.

(g) Under what conditions does first order perturbation theory give an accurate result for $P_n(t)$? Find a sufficient condition on the impulse $\Delta p(t) \equiv \int_0^t dt' F(t')$, for the first order result to be accurate. Express your condition in terms of $\Delta p(t)$, x_0, and $\hbar$.