HW#4—Phys623—Spring 2017

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1. The structure of nuclei can be approximately described using the so-called *shell model*. Interacting nucleons (protons and neutrons) produce a self-consistent, spherically symmetric field of the nuclear force. The energy levels of a nucleon in this field can be classified by the values of the orbital angular momentum, the radial quantum number, and the total angular momentum of the nucleon. This is somewhat similar to the classification of electron energy levels in an atom. The neutrons and protons are fermions and therefore each obey the Pauli exclusion principle. The nuclei with completely filled energy shells—"(doubly) magic nuclei"—are particularly stable, similarly to the noble elements with completely filled electron shells. In addition to the common nuclear potential, each nucleon has a spin-orbit coupling $-2a\mathbf{L}\cdot\mathbf{S}$, where a is a positive constant.

(a) How many nucleons can be placed in the lowest $1S$ shell of a nucleus? What is the name of this particle?

(b) The first two levels of the nuclear potential are $1S$, $1P$ (where "1" stands for the radial quantum number.) What is the second magic nucleus? Explain how the counting goes.

(c) Use the shell model with spin-orbit coupling to predict the spin and parity of the nuclei

$$^{2}\mathrm{H},\quad ^{3}\mathrm{H},\quad ^{7}\mathrm{Li},\quad ^{11}\mathrm{B},\quad ^{15}\mathrm{N}$$

(where the subscript denotes the number of nucleons, i.e. protons plus neutrons). Check your prediction by looking up the answer. For ^{2}H there is more than one possibility, so list them all, and then select one by using the fact that, as a result of the "tensor force", the neutron and proton form a state that is symmetric under n-p interchange. For ^{7}Li there is also more than one possibility, so list them all, and then select one by using the fact that the neutrons in the outer shell form a $J=0$ pair.

2. Find an upper bound for the ground state energy of the hydrogen atom using a three-dimensional harmonic oscillator ground state wave function

$$\psi(r) = (\sqrt{\pi}\beta)^{-3/2}\exp(-r^2/2\beta^2)$$

as a trial function. Compare with the true ground state energy. (See section 11.2 of Schwabl, or Notes 27 of Littlejohn for discussion of the *Variational method.*)

3. Schwabl, Problem 13.6 (*Virial theorem for atoms*)
[*Hint 2*: Show, and use, the following fact: If $|\psi\rangle$ is any eigenvector of a Hamiltonian H (not necessarily the ground state), and $|\psi(\beta)\rangle$ is a 1-parameter family of normalized states, with $|\psi(0)\rangle = |\psi\rangle$, then $(d/d\beta)\langle\psi(\beta)|H|\psi(\beta)\rangle|_{\beta=0} = 0$.]

4. **(a)** Consider a quantum particle in one dimension, in a potential satisfying $V(x) = 0$ for $|x| > R$ for some R, and $\int V(x)\,dx < 0$ (i.e. the average potential is negative). Use the variational principle to establish the existence of a bound state for this potential, using a Gaussian trial wavefunction

$$\psi(r) = (\sqrt{\pi}\beta)^{-1/2} \exp(-x^2/2\beta^2)$$

with the width β as the variational parameter. [*Hint*: Examine what happens to the kinetic and potential energy when $\beta \to \infty$. In fact, this could be done essentially using dimensional analysis.]

(b) Now consider the corresponding question in $d \geq 2$ spatial dimensions. Show that the $d = 1$ argument fails to generalize.

5. *Toy model of a helium atom*, Qualifier, January 1999, II-2.

Two particles, each of mass m, are confined in one dimension to a box of length L.

(a) First consider the case where the particles are spinless, not identical, and do not interact between themselves.

What are the normalized two-particle wave functions and energies of the three lowest-energy states of the system? Are any of these energy levels degenerate?

(b) Suppose the particles interact between themselves with the potential $V = \lambda\delta(x_1 - x_2)$, where x_1 and x_2 are the coordinates of the particles, $\delta(x)$ is the one-dimensional Dirac delta-function, and $\lambda > 0$.

In the lowest order of perturbation theory in V calculate the energies and two-particle wave functions for the three lowest-energy states of the system. Sketch how the energy levels shift relative to the energy levels of the noninteracting system.

(c) Formulate a condition on the coefficient λ for lowest order perturbation theory to be applicable.

(d) Now suppose that the particles are two identical fermions, each of spin 1/2, interacting via the potential V of part 5b. Explain how the Pauli principle determines the values of the total spin of the system for the three energy levels found in part 5b. Write the values of the total spin and the degeneracy next to the energy levels in the diagram of part 5b.

Possibly useful integrals:

$$\int_0^\pi d\phi \, \sin^4\phi = 3\pi/8 \qquad\qquad \int_0^\pi d\phi \, \sin^2\phi \, \sin^2 2\phi = \pi/4$$