HW#2—Phys623—Spring 2017

Prof. Ted Jacobson
PSC 3151, (301)405-6020
jacobson@physics.umd.edu

1. Analyze the effect of an external electric field on the $2s_{1/2}$ and $2p_{1/2}$ levels of hydrogen, taking into account that these levels are split by the Lamb shift by an amount 1.06 GHz. The other levels, and the hyperfine structure, can be ignored for the purpose of this problem, so this is only a two-level system. Show that the energy shifts are quadratic in the field strength for small amplitude electric fields, but that they become linear at larger strengths. Thus there is a threshold field strength, call it $\mathcal{E}_0$, where the behavior changes from quadratic to linear. Estimate $\mathcal{E}_0$ in Volts/cm. Find and sketch the energy levels as a function of the electric field strength $\mathcal{E}$. Explain how this problem provides an example of Kramers degeneracy (and use this fact to cut your work in half). In computing the required matrix elements, you may construct the $2s_{1/2}$ and $2p_{1/2}$ states using the nonrelativistic, hydrogen wave functions, for which $\langle 2p, m_\ell = 0|z|2s\rangle = 3a_0$.

2. Write the $2p_{1/2}(F = 0)$ state of hydrogen in terms of the product kets $|2p, m_l\rangle|m_s\rangle|m_I\rangle$. (Here $F = J + I$ is the total angular momentum of the atom, where I is the nuclear spin and J is the total angular momentum of the electron.) Do this by two methods, and compare: i) first find the $J = 1/2$ states of the electron, and then combine with the nuclear spin states to find the $F = 0$ state; ii) list all the product kets in the L, S, I basis that have $m_F = 0$, and then determine the coefficients of the unique (normalized) superposition of these that is invariant under all rotations. (*Hint*: Use F_+.)

3. The ground state of the hydrogen atom is split into two hyperfine states separated by 1.42 GHz. What is the hyperfine splitting in the deuterium atom? The respective magnetic moments are $\mu = 2.8\mu_N$ and $\mu_d = 0.86\mu_N$, where μ_N is the nuclear magneton.

4. The Dirac equation implies that the g-factor for the electron is $g_e = 2$. This result can also be obtained from the nonrelativistic limit of the Dirac Hamiltonian. For a charge e coupled to an electromagnetic vector potential $\vec{A}(x)$ this nonrelativistic Hamiltonian for the two-component wave function is $H = [\vec{\sigma} \cdot (\vec{p} - \frac{e}{c}\vec{A})]^2/2m$. Show that this implies $g_e = 2$. (*Hint*: It is convenient to use index notation, and $\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$.)

5. The Weyl equation for massless, spin-1/2 particles is

$$i\partial_t\chi = \pm\vec{\sigma} \cdot \vec{p}\,\chi,$$

where $\vec{\sigma}$ is the vector of Pauli matrices, χ is a two-component spinor, and $\vec{p}$ is the usual quantum mechanical momentum operator. It is the time evolution equation for a quantum system with Hamiltonian $H = \pm\vec{\sigma} \cdot \vec{p}$. The two possible signs correspond to the "chirality" of the massless spin-1/2 particle, called right-handed $(+)$ and left-handed $(-)$.

(a) Use dimensional analysis to restore the factors of $\hbar$ and c in the Weyl equation (but after this part set $\hbar = c = 1$).

(b) Establish the following:

i. The Hamiltonian is translation invariant, and momentum is conserved.

ii. The Hamiltonian is invariant under time reversal, but not under parity (space inversion).

iii. The "helicity" $\frac{1}{2}\vec{\sigma}\cdot\hat{p}$ (spin along the direction of momentum) is conserved in time.

iv. An eigenstate of momentum and helicity is an energy eigenstate with energy E satisfying $E = \pm|\vec{p}|$. Energy eigenstates of right (left)-handed Weyl particles with positive (negative) helicity have positive energy.

v. The Heisenberg spin operator $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$ is not conserved. Rather it satisfies $d\vec{S}/dt = \mp 2\vec{p}\times\vec{S}$.

vi. The Heisenberg velocity operator $d\vec{x}/dt$ is $\pm\vec{\sigma}$, which is not conserved.

(c) The Dirac equation consists of a pair of Weyl equations, for right and left chirality spinors R and L, coupled by terms that transform R into L and vice versa:

$$i\partial_t R = \vec{\sigma}\cdot\vec{p}\,R + mL, \qquad i\partial_t L = -\vec{\sigma}\cdot\vec{p}\,L + mR.$$

i. Show that $H^2 = p^2 + m^2$, where H is the full Hamiltonian acting on the 4-component wave function $\begin{pmatrix} R \\ L \end{pmatrix}$. This means that an energy and momentum eigenstate satisfies $E^2 = p^2 + m^2$, so that m is evidently the mass of the particle.

ii. Find the positive energy eigenstates with zero momentum. How do they transform under parity?