HW#11—Phys623—Spring 2017

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1. Reconcile these two statements: (i) The optical theorem states that the total cross section is $4\pi/k$ times the imaginary part of the forward scattering amplitude. (ii) In the first Born approximation, the forward scattering amplitude is real.

2. *Bell states* form a basis for the the four dimensional Hilbert space of two-qubit states,

$$|\phi^{\pm}\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}, \qquad |\psi^{\pm}\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}. \tag{1}$$

(a) Show that these states are "maximally entangled", in the sense that the reduced density matrix on either bit is maximally mixed. (b) Find unitary transformations, acting on the first qubit, that turn $|\phi^+\rangle$ into $|\phi^-\rangle$, $|\psi^+\rangle$, and $|\psi^-\rangle$, respectively. (Similarly, any of the four states can be mapped to any other by such a unitary transformation.)

3. Let two spins-1/2 interact, with Hamiltonian $H = \alpha \vec{S}_1 \cdot \vec{S}_2$, and suppose at $t = 0$ their state is $|\uparrow\downarrow\rangle$. (a) Find the reduced density matrix for the first spin as a function of time. Make sure it is hermitian, its trace is unity, and its eigenvalues lie between 0 and 1. (b) Is there a time at which the first spin is certain to be in the state $|\downarrow\rangle$? If so, what is that time? (c) Is there a time when the spins are maximally entangled? If so, what is that time?

4. *An example of quantum erasure correction*

Establish the validity of the statement below typeset in blue.

Text taken verbatim from *"Bulk Locality and Quantum Error Correction in AdS/CFT,"* by Ahmed Almheiri, Xi Dong, and Daniel Harlow.

The simplest example of quantum error correction actually involves three-state "qutrits" instead of two-state qubits, and it uses three qutrits to send a single-qutrit message. Say Alice wishes to send the state

$$|\psi\rangle = \sum_{i=0}^{2} a_i |i\rangle. \tag{2}$$

The idea is to instead send the state

$$|\widetilde{\psi}\rangle = \sum_{i=0}^{2} a_i |\widetilde{i}\rangle, \tag{3}$$

where

$$\begin{aligned}
|\widetilde{0}\rangle &= \frac{1}{\sqrt{3}}\left(|000\rangle + |111\rangle + |222\rangle\right) \\
|\widetilde{1}\rangle &= \frac{1}{\sqrt{3}}\left(|012\rangle + |120\rangle + |201\rangle\right) \\
|\widetilde{2}\rangle &= \frac{1}{\sqrt{3}}\left(|021\rangle + |102\rangle + |210\rangle\right).
\end{aligned} \tag{4}$$

This protocol has two remarkable properties. First of all for any state $|\widetilde{\psi}\rangle$, the reduced density matrix on any one of the qutrits is maximally mixed. Thus no single qutrit can be used to acquire any information about the state. Secondly, from any two of the qutrits Bob can reconstruct the state. For example, say he has access to only the first two qutrits. He can make use of the fact that there exists a unitary transformation U_{12} acting only on the first two qutrits that implements

$$(U_{12} \otimes I_3) \, |\widetilde{i}\rangle = |i\rangle \otimes \frac{1}{\sqrt{3}} \left(|00\rangle + |11\rangle + |22\rangle \right). \tag{5}$$

Acting with this on the encoded message, we see that Bob can recover the state $|\psi\rangle$:

$$(U_{12} \otimes I_3) \, |\widetilde{\psi}\rangle = |\psi\rangle \otimes \frac{1}{\sqrt{3}} \left(|00\rangle + |11\rangle + |22\rangle \right). \tag{6}$$

Explicitly U_{12} is a permutation that acts as

$$\begin{array}{lll} |00\rangle \to |00\rangle & |11\rangle \to |01\rangle & |22\rangle \to |02\rangle \\ |01\rangle \to |12\rangle & |12\rangle \to |10\rangle & |20\rangle \to |11\rangle \\ |02\rangle \to |21\rangle & |10\rangle \to |22\rangle & |21\rangle \to |20\rangle \end{array} . \tag{7}$$

Clearly by the symmetry of (4) a similar construction is also possible if Bob has access only to the second and third, or first and third qutrits. Thus Bob can correct for the loss of any one of the qutrits; in quantum information terminology one describes this as a quantum error correcting code that can protect against arbitrary single qutrit erasures. The subspace spanned by (4) is called the *code subspace*; the entanglement of the states in the code subspace is essential for the functioning of the protocol.