

1. Problem 11.1, *Schwabl*. (Use units with $\hbar = m = \omega = 1$, and use Schwabl's hint.) To reduce the labor, *solve this problem only for the ground state*. Add parts: (b) Find the first order perturbation to the ground state vector. (c) Find the first order perturbation to the expectation value of x in the ground state, and compare with the minimum of the potential function (which you can find perturbatively).
2. Problem 11.3, *Schwabl*. (Use the x and y ladder operators. N.B. also $\omega = 1$.)
3. Consider the Hamiltonian $H = H_0 + V$ of a three state system, with

$$H_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & v \\ 0 & 0 & v \\ v & v & 0 \end{pmatrix}. \quad (1)$$

As you can check by hand, the eigenvalues are $0, \lambda_{\pm}$, with corresponding normalized eigenvectors $(1, -1, 0)/\sqrt{2}$ and $a_{\pm}(v, v, \lambda_{\pm})$, where $\lambda_{\pm} = (1 \pm \sqrt{1 + 8v^2})/2$ and $a_{\pm} = (2v^2 + \lambda_{\pm}^2)^{-1/2}$. Expanding in small v , and keeping up to quadratic terms in v , one can (carefully) show that the nonzero eigenvalues and corresponding (normalized to $O(v^2)$) eigenvectors become

$$\lambda_+ = 1 + 2v^2, \quad (v, v, 1 - v^2) \quad (2)$$

$$\lambda_- = -2v^2, \quad \text{sgn}(v)((1 - v^2), (1 - v^2), -2v)/\sqrt{2} \quad (3)$$

- (a) Use degenerate perturbation theory (with V as the perturbation) to find the first and second order energy shifts $E^{(1)}$ and $E^{(2)}$ for the three eigenvalues, and check that they agree with the expansions given above.
 - (b) Find the eigenvectors of the second order secular equation and compare with the $v \rightarrow 0$ limit of the exact eigenvectors. They should agree.
4. Consider a two-state quantum system described by the Hamiltonian

$$H = \begin{pmatrix} E + U & \Delta e^{i\phi} \\ \Delta e^{-i\phi} & E - U \end{pmatrix}, \quad (4)$$

with E, U, Δ , and ϕ all real. This is the most general hermitian 2×2 matrix.

- (a) Find the exact eigenvalues and eigenvectors of H . (*Tip*: Expand H in Pauli matrices and use what you know about them.)
- (b) Sketch the eigenvalues as functions of U when U changes from $U \ll -\Delta$ to $U \gg \Delta$. Notice that the energy levels “repel” in the region $U \approx 0$ where they would cross if Δ were zero.
- (c) Expand the exact eigenvalues to lowest nonvanishing order in U/Δ when $U \ll \Delta$.

- (d) Considering the Δ terms of the Hamiltonian (4) as a perturbation, compute the first and second order energy level shifts using non-degenerate perturbation theory (assume $U \neq 0$.)
 - (e) Compute the first order correction to the energy eigenstates.
 - (f) The approximate eigenvalues of parts (4c) and (4d) do not agree when $0 < U \ll \Delta$. Explain why non-degenerate perturbation theory does not give good results even though the unperturbed eigenvalues are non-degenerate when $U \neq 0$.
5. In most calculations of atomic energy levels the nucleus is taken as a positive point charge Ze . Actually, the nuclear charge is more accurately represented by a uniform charge distribution reaching to a radius of about $Z^{1/3}$ Fermi. (1 Fermi = 10^{-13} cm = 2×10^{-5} Bohr radius.)
- (a) Use first order perturbation theory to calculate the correction to the energy of a 1s electron due to this nuclear size effect. The perturbation ΔV is the difference of the true potential and the Coulomb potential inside the nucleus. How does the correction depend on the nuclear charge Z ?
 - (b) Why is the shift for the 2p state negligible compared to that for the 1s state?
 - (c) ΔV blows up at the origin, so you might think that perturbation theory won't be accurate. However, what really matters is how much the state changes. Estimate the norm of the perturbation of the state, to assess whether or not perturbation theory should be accurate.

Notes: (i) You can simplify the calculation by noting that the nucleus is much smaller than the Bohr radius, so the wave function is approximately constant inside the nucleus. (ii) Later we'll look into whether it's accurate to neglect relativistic corrections in this calculation. This question is, surprisingly, related to the proton radius puzzle (https://en.wikipedia.org/wiki/Proton_radius_puzzle), for which Cliff Burgess (an old friend from grad school) very recently found the solution (<https://arxiv.org/abs/1612.07337>).