Comment on "Charge density on a conducting needle," by David J. Griffiths and Ye Li [Am. J. Phys. 64(6), 706–714 (1996)]

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Griffiths and Li[1] suggest with some reluctance that the linear charge density on a conducting needle might be "in fact a *constant.*" The purpose of this Comment is to support that conjecture with a geometrical demonstration.

We shall begin with the well-known proof that the electrostatic field inside a uniformly charged spherical shell is zero. Pick an arbitrary point, and draw a narrow cone (in both directions). The charge Q is proportional to the area subtended, which is proportional to the square of the distance r; so for the two areas at opposite ends of the cone:

$$\frac{Q_1}{r_1^2}=\frac{Q_2}{r_2^2}, \tag{1}$$

which is the same as the condition that the E fields from the two charges are equal and opposite at the point. (It works only for a sphere because an arbitrary straight line hits opposite sides at the same angle to the spherical surface.) The whole sphere can be mapped into such mutually canceling pairs for any interior point. Therefore, the E field is zero inside the sphere.

Building on this concept, we approach the conducting circular disk. R. Friedberg[2] suggests (p. 1087): "Imagine a sphere of radius a centered at the origin, and on its surface place a uniform surface charge density $\sigma_0=Q/4\pi a^2$. Now collapse the z direction so that the charge is projected vertically onto the $x-y$ plane. This gives a surface density $\sigma(x,y)$ on the disk... ." Referring to Fig. 1, for a point on the disk,

$$\frac{x_1}{r_1}=\frac{x_2}{r_2} \tag{2}$$

(similar triangles). So, from Eq. (1),

$$\frac{Q_1}{x_1^2}=\frac{Q_2}{x_2^2} \tag{3}$$

and, in general, the x and y components of the E field cancel out as above, which means that the disk is an equipotential, as required for a conductor in electrostatics.

The expression for the surface charge density follows immediately. For $\sigma(x,y)$ we can write $\sigma(\rho)$, with $\rho = \sqrt{x^2+y^2}$ (see Fig. 2). The projection from the two hemispheres to the disk gives $\sigma(\rho)=2\sigma_0/\sin\theta$, where $\cos\theta=\rho/a$; so

$$\sigma(\rho)=2\sigma_0\frac{a}{\sqrt{a^2-\rho^2}} \tag{4}$$

(which increases without limit at the edge of the disk).

Let us apply a similar procedure to the conducting needle, Fig. 3. Now, instead of projecting down to a plane, we are projecting the charge onto a line along the x axis. As before, the E field component along the line is zero, so the resulting

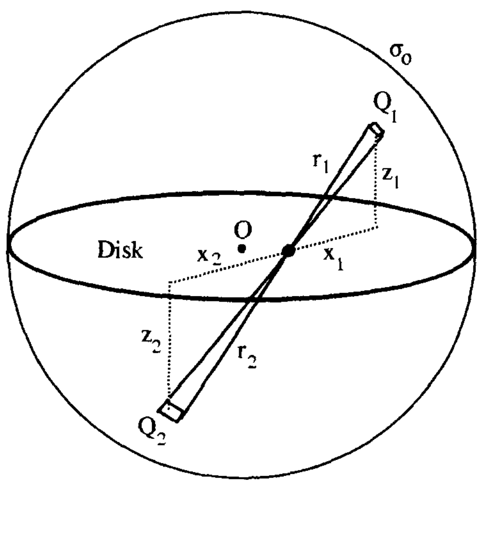

Fig. 1. Equipotential disk.

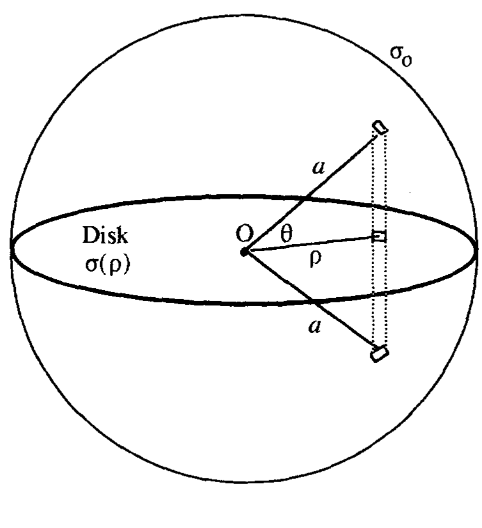

Fig. 2. Projection onto a disk.

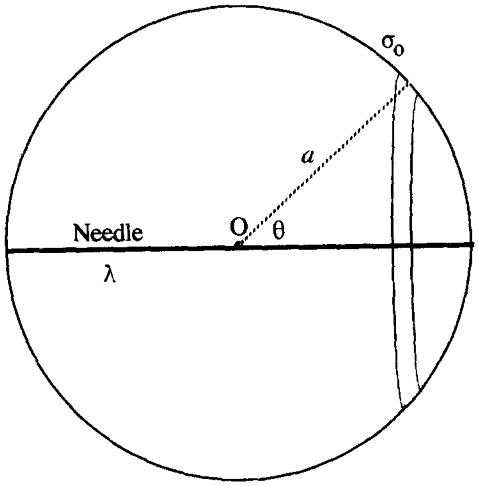

Fig. 3. Projection onto a line.

linear charge density λ is appropriate for a conductor. In order to find λ, we project a ring of charge from the spherical surface onto the corresponding line element of the needle:

$$\sigma_0 2\pi a \sin\theta\, a d\theta = \lambda a d\theta \sin\theta. \tag{5}$$

So the linear charge density turns out to be a constant,

$$\lambda = 2\pi a \sigma_0 = \frac{Q}{2a} \tag{6}$$

corresponding to "the counterintuitive possibility" suggested by Griffiths and Li.

It is always possible to carve up a uniform line charge in such a way as to show that the field at an arbitrary point within the line is zero. A numerical example will probably be most understandable, and most convincing, Fig. 4. For clarity, we measure x_1 from the origin toward the right, and x_2 toward the left. What we need is that, for an arbitrary origin 0, any given segment dx_1 is canceled by the corresponding segment dx_2, in terms of field at the origin:

$$\frac{dx_1}{x_1^2} = \frac{dx_2}{x_2^2}, \tag{7}$$

since dx is proportional to charge. The solution is

$$\frac{1}{x_1} = \frac{1}{x_2} + C. \tag{8}$$

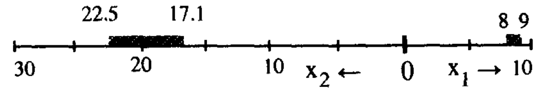

Fig. 4. Apportionment of a line charge.

For the example given, $C = 1/10 - 1/30 = 2/30 = 0.0667$. Then we find that $x_1 = 8$ corresponds to $x_2 = 17.1$, and $x_1 = 9$ to $x_2 = 22.5$. Taking averages of these distances, if the two segments cancel out then

$$\frac{5.4}{19.8^2} \approx \frac{1}{8.5^2}, \tag{9}$$

$$0.0138 = 0.0138, \tag{10}$$

so that checks pretty well. For any given segment on the right of the origin, the corresponding one on the left produces a field that is equal and opposite.

And that's all very plausible, except for the following paradox, alluded to in Griffiths and Li's conclusion. Referring to Fig. 4, symmetry would seem to dictate that the E field at the origin 0 due to the charge on the right should be exactly equal to the field due to the charge just up to the "10" mark on the left, and so the remaining charge on the left should provide a net field pointing to the right (for positive λ), in contradiction to our conclusion that there is no field component along the needle. This might be called a "paradox of infinity" which produces what Griffiths and Li refer to as a "deep pathology in the reduction to a one-dimensional object that does not infect the reduction from three to two." The E field and scalar potential V are infinite for the needle, and the capacitance is zero, none of which is true for the disk (except for E at the very edge). In particular, in Fig. 4, the field at the origin due to the charge on the right is infinite, and the field from the left is also infinite; so adding a finite field due to the charge from 10 to 30 on the left will make only a negligible difference to the situation at that point. This is presumably a satisfactory approximation for a physically realizable thin needle.

So, in view of the above and Ref. 1, it appears that the linear charge density is uniform for a needle of infinitesimal thickness and for a thin prolate spheroid; and that, for a cylinder of finite girth, there is an excess of charge at the ends.

[1]D. J. Griffiths and Y. Li, "Charge density on a conducting needle," Am. J. Phys. **64**, 706–714 (1996).

[2]R. Friedberg, "The electrostatics and magnetostatics of a conducting disk," Am. J. Phys. **61**, 1084–1096 (1993).

RADIO ENGINEERING

Wally [Selove] was a superb student in high school, and the University of Chicago offered him a tuition scholarship. The university awakened his intellect. He had thought he wanted to be a radio engineer but Chicago has no such major. He was advised that physics was the closest area. Wally sought out extra work to do in his classes because he needed an A average to maintain his scholarship. He found physics to be much more interesting than he had imagined, and he was fascinated by marvelous courses in the classics and in history.

Fay Ajzenberg-Selove, *A Matter of Choices—Memoirs of a Female Physicist* (Rutgers University Press, New Brunswick, New Jersey, 1994), p. 113.