

Magnetic monopoles

If electromagnetism is embedded into a “larger” theory that contains other fields, such as a “grand unified theory”, there may exist field configurations that produce an extended object that has a magnetic monopole field. The magnetic charge of such objects is quantized — it must be an integer multiple of $h/e\mu_0$, where h is Planck’s constant. In this problem you will (sort of) show why.

1. Inside the object, $\nabla \cdot \mathbf{B}$ is nonzero, and equal to some other field quantity, but outside the object we have ordinary electromagnetism, hence $\nabla \cdot \mathbf{B} = 0$. Consider a sphere surrounding the monopole, outside the core. Everywhere on the sphere $\nabla \cdot \mathbf{B} = 0$, so $\mathbf{B} = \nabla \times \mathbf{A}$ for some vector potential \mathbf{A} . This seems to imply the flux of through the sphere must vanish. Using Stokes’ theorem, show that the flux through the northern hemisphere is equal to the line integral of \mathbf{A} around the equator one way, while the flux through the southern hemisphere is equal to the line integral of \mathbf{A} around the equator the other way.
2. When you add these together, they cancel, so the total flux must be zero! There is an escape from this conclusion however: the existence of a vector potential for a divergenceless vector field is only guaranteed in a region of space that has no holes in it. Since $\text{div } \mathbf{B}$ is not zero inside the monopole core, the core is a “hole” in the region. But the escape is not so complete, since we can always find a vector potential \mathbf{A}_N that works just for the northern hemisphere, and another \mathbf{A}_S that works just for the southern hemisphere. Show that the total flux through the sphere is equal to

$$\oint_{\text{equator}} (\mathbf{A}_N - \mathbf{A}_S) \cdot d\mathbf{l}.$$

3. The difference $\mathbf{A}_N - \mathbf{A}_S$ cannot be arbitrary: both vector potentials must give the same magnetic field on the equator. Hence, on the equator, the two vector potentials must be related by a gauge transformation, $\mathbf{A}_N - \mathbf{A}_S = \nabla\lambda$. Show that

$$\oint_{\text{equator}} \nabla\lambda \cdot d\mathbf{l} = \lambda(2\pi) - \lambda(0).$$

4. The total magnetic flux through the sphere is therefore equal to the discontinuity in λ at $\phi = 0, 2\pi$ on the equator. Since λ is only a gauge parameter, you might think it is entirely unphysical, and hence that it could have an arbitrary discontinuity. In quantum mechanics, however, this discontinuity cannot be arbitrary. The reason is that Schrodinger’s equation for charged particles in a magnetic field is unchanged

when a gauge transformation $\mathbf{A} \rightarrow \mathbf{A} + \nabla\lambda$ is made on the vector potential only if the wave function of a charge e is simultaneously changed by a phase factor $\psi \rightarrow e^{ie\lambda/\hbar}\psi$. This phase change doesn't alter the probability density $\psi^*\psi$, so it leaves the physical meaning of the wave-function intact. However, the wave-function itself must be continuous. Show that continuity of the wave-function requires that the jump in $e\lambda/\hbar$ must be an integer multiple of 2π .

5. Show that this reasoning implies that the magnetic monopole charge must be an integer multiple of $h/e\mu_0$, where $\hbar = h/2\pi$.¹
6. Use the Aharonov-Bohm effect problem of this homework set to show that the monopole charge quantization condition is equivalent to the condition that the field in a solenoid threaded by the same flux as the monopole flux would not be seen by quantum particles with charge e .

This is just how Dirac originally derived the monopole charge quantization condition: He imagined a monopole as an open endpoint of an infinitely thin, "invisible" solenoid, the so-called "Dirac string". He wasn't thinking about embedding electromagnetism into a larger theory, rather he just noticed that quantum electrodynamics, as it is, seems to admit these quantized, point magnetic charges.

¹I've assumed here that the unit of magnetic charge is defined such that the field of a monopole of magnetic charge Q_m is $(\mu_0/4\pi)Q_m/r^2$. I think this choice is required if the magnetic dipole moment of a $\pm Q_m$ monopole pair separated by a distance d is to be $Q_m d$. Alternatively, for example, one could omit the μ_0 from the monopole field, but then the dipole moment of the pair would be $Q_m d/\mu_0$.