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Calculus of Variations I

The Lagrangian Formalism

With 73 Figures

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Introduction

The Calculus of Variations is the art to find optimal solutions and to describe their essential properties. In daily life one has regularly to decide such questions as which solution of a problem is best or worst; which object has some property to a highest or lowest degree; what is the optimal strategy to reach some goal. For example one might ask what is the shortest way from one point to another, or the quickest connection of two points in a certain situation. The isoperimetric problem, already considered in antiquity, is another question of this kind. Here one has the task to find among all closed curves of a given length the one enclosing maximal area. The appeal of such optimum problems consists in the fact that, usually, they are easy to formulate and to understand, but much less easy to solve. For this reason the calculus of variations or, as it was called in earlier days, the isoperimetric method has been a thriving force in the development of analysis and geometry.

An ideal shared by most craftsmen, artists, engineers, and scientists is the principle of the economy of means: What you can do, you can do simply. This aesthetic concept also suggests the idea that nature proceeds in the simplest, the most efficient way. Newton wrote in his *Principia*: "*Nature does nothing in vain, and more is in vain when less will serve; for Nature is pleased with simplicity and affects not the pomp of superfluous causes.*" Thus it is not surprising that from the very beginning of modern science optimum principles were used to formulate the "laws of nature", be it that such principles particularly appeal to scientists striving toward unification and simplification of knowledge, or that they seem to reflect the preestablished harmony of our universe. Euler wrote in his *Methodus inveniendi* [2] from 1744, the first treatise on the calculus of variations: "*Because the shape of the whole universe is most perfect and, in fact, designed by the wisest creator, nothing in all of the world will occur in which no maximum or minimum rule is somehow shining forth.*" Our belief in the best of all possible worlds and its preestablished harmony claimed by Leibniz might now be shaken; yet there remains the fact that many if not all laws of nature can be given the form of an extremal principle.

The first known principle of this type is due to Heron from Alexandria (about 100 A.D.) who explained the law of reflection of light rays by the postulate that *light must always take the shortest path.* In 1662 Fermat succeeded in deriving the law of refraction of light from the hypothesis that *light always propagates in the quickest way from one point to another.* This assumption is now called *Fermat's principle*. It is one of the pillars on which geometric optics rests; the other one is *Huygens's principle* which was formulated about 15 years later. Further, in his letter to De la Chambre from January 1, 1662, Fermat motivated his principle by the following remark: "*La nature agit toujour par les voies les plus courtes.*" (Nature always acts in the shortest way.)

About 80 years later Maupertuis, by then President of the Prussian Academy of Sciences, resumed Fermat's idea and postulated his metaphysical principle of the *parsimonious universe*, which later became known as "*principle of least action*" or "*Maupertuis's principle*". He stated: *If there occurs some change in nature, the amount of action necessary for this change must be as small as possible.*

"Action" that nature is supposed to consume so thriftily is a quantity introduced by Leibniz which has the dimension "energy × time". It is exactly that quantity which, according to Planck's quantum principle (1900), comes in integer multiples of the elementary quantum h.

In the writings of Maupertuis the action principle remained somewhat vague and not very convincing, and by Voltaire's attacks it was mercilessly ridiculed. This might be one of the reasons why Lagrange founded his *Méchanique analitique* from 1788 on d'Alembert's principle and not on the least action principle, although he possessed a fairly general mathematical formulation of it already in 1760. Much later Hamilton and Jacobi formulated quite satisfactory versions of the action principle for point mechanics, and eventually Helmholtz raised it to the rank of the most general law of physics. In the first half of this century physicists seemed to prefer the formulation of natural laws in terms of space–time differential equations, but recently the principle of least action had a remarkable comeback as it easily lends itself to a global, coordinate-free setup of physical "field theories" and to symmetry considerations.

The development of the calculus of variations began briefly after the invention of the infinitesimal calculus. The first problem gaining international fame, known as "problem of quickest descent" or as "brachystochrone problem", was posed by Johann Bernoulli in 1696. He and his older brother Jakob Bernoulli are the true founders of the new field, although also Leibniz, Newton, Huygens and l'Hospital added important contributions. In the hands of Euler and Lagrange the calculus of variations became a flexible and efficient theory applicable to a multitude of physical and geometric problems. Lagrange invented the δ-calculus which he viewed to be a kind of "higher" infinitesimal calculus, and Euler showed that the δ-calculus can be reduced to the ordinary infinitesimal calculus. Euler also invented the multiplier method, and he was the first to treat variational problems with differential equations as subsidiary conditions. The development of the calculus of variations in the 18th century is described in the booklet by Woodhouse [1] from 1810 and in the first three chapters of H.H. Goldstine's historical treatise [1]. In this first period the variational calculus was essentially concerned with deriving necessary conditions such as Euler's equations which are to be satisfied by minimizers or maximizers of variational problems. Euler mostly treated variational problems for single integrals where the corresponding Euler equations are ordinary differential equations, which he solved in many cases by very skillful and intricate integration techniques. The spirit of this development is reflected in the first parts of this volume. To be fair with Euler's achievements we have to emphasize that he treated in [2] many more one-dimensional variational problems than the reader can find anywhere else including our book, some of which are quite involved even for a mathematician of today.

However, no *sufficient conditions* ensuring the minimum property of solutions of Euler's equations were given in this period, with the single exception of a paper by Johann Bernoulli from 1718 which remained unnoticed for about 200 years. This is to say, analysts were only concerned with determining solutions of Euler equations, that is, with stationary curves of one-dimensional variational problems, while it was more or less taken for granted that such stationary objects furnish a real extremum.

The sufficiency question was for the first time systematically tackled in Legendre's paper [1] from 1788. Here Legendre used the idea to study the second variation of a functional for deciding such questions. Legendre's paper contained some errors, pointed out by Lagrange in 1797, but his ideas proved to be fruitful when Jacobi resumed the question in 1837. In his short paper [1] he sketched an entire theory of the second variation including his celebrated theory of conjugate points, but all of his results were stated with essentially no proofs. It took a whole generation of mathematicians to fill in the details. We have described the basic features of the *Legendre–Jacobi theory* of the second variation in Chapters 4 and 5 of this volume.

Euler treated only a few variational problems involving multiple integrals. Lagrange derived the "Euler equations" for double integrals, i.e. the necessary differential equations to be satisfied by minimizers or maximizers. For example he stated the minimal surface equation which characterizes the stationary surface of the nonparametric area integral. However he did not indicate how one can obtain solutions of the minimal surface equation or of any other related Euler equation. Moreover neither he nor anyone else of his time was able to derive the *natural boundary conditions* to be satisfied by, say, minimizers of a double integral subject to free boundary conditions since the tool of "integration by parts" was not available. The first to successfully tackle two-dimensional variational problems with free boundaries was Gauss in his paper [3] from 1830 where he established a variational theory of capillary phenomena based on Johann Bernoulli's *principle of virtual work* from 1717. This principle states that in equilibrium no work is needed to achieve an infinitesimal displacement of a mechanical system. Using the concept of a potential energy which is thought to be attached to any state of a physical system, Bernoulli's principle can be replaced by the following hypothesis, *the principle of minimal energy*: The equilibrium states of a physical system are stationary states of its potential energy, and the stable equilibrium states minimize energy among all other "virtual" states which lie close-by.

For capillary surfaces not subject to any gravitational forces the potential energy is proportional to their surface area. This explains why the phenomenological theory of soap films is just the theory of surfaces of minimal area.

After Gauss free boundary problems were considered by Poisson, Ostrogradski, Delaunay, Sarrus, and Cauchy. In 1842 the French Academy proposed as topic for their great mathematical prize the problem to derive the natural boundary conditions which together with Euler's equations must be satisfied by minimizers and maximizers of free boundary value problems for multiple integrals. Four papers were sent in; the prize went to Sarrus with an honourable mentioning of Delaunay, and in 1861 Todhunter [1] held Sarrus's paper for "the most important original contribution to the calculus of variations which has been made during the present century". It is hard to believe that these formulas which can nowadays be derived in a few lines were so highly appreciated by the Academy, but we must realize that in those days integration by parts was not a fully developed tool. This example shows very well how the problems posed by the variational calculus forced analysts to develop new tools. Time and again we find similar examples in the history of this field.

In Chapters 1–4 we have presented all formal aspects of the calculus of variations including all necessary conditions. We have simultaneously treated extrema of single and multiple integrals as there is barely any difference in the degree of difficulty, at least as long as one sticks to variational problems involving only first order derivatives. The difference between one- and multi-dimensional problems is rarely visible in the formal aspect of the theory but becomes only perceptible when one really wants to construct solutions. This is due to the fact that the necessary conditions for one-dimensional integrals are ordinary differential equations, whereas the Euler equations for multiple integrals are partial differential equations. The problem to solve such equations under prescribed boundary conditions is a much more difficult task than the corresponding problem for ordinary differential equations; except for some special cases it was only solved in this century. As we need rather refined tools of analysis to tackle partial differential equations we deal here only with the formal aspects of the calculus of variations in full generality while existence questions are merely studied for one-dimensional variational problems. The existence and regularity theory of multiple variational integrals will be treated in a separate treatise.

Scheeffer and Weierstrass discovered that positivity of the second variation at a stationary curve is not enough to ensure that the curve furnishes a local minimum; in general one can only show that it is a *weak minimizer*. This means that the curve yields a minimum only in comparison to those curves whose tangents are not much different.

In 1879 Weierstrass discovered a method which enables one to establish a *strong minimum property* for solutions of Euler's equations, i.e. for stationary curves; this method has become known as *Weierstrass field theory*. In essence Weierstrass's method is a rather subtle convexity argument which uses two ingredients. First one employs a local convexity assumption on the integrand of the variational integral which is formulated by means of *Weierstrass's excess function*. Secondly, to make proper use of this assumption one has to embed the given stationary curve in a suitable field of such curves. This field embedding can be interpreted as an introduction of a particular system of normal coordinates which very much simplify the comparison of the given stationary curve with any neighbouring curve. In the plane it suffices to embed the given curve in an arbitrarily chosen field of stationary curves while in higher dimensions one has to embed the curve in a so-called *Mayer field*.

In Chapter 6 of this volume we shall describe Weierstrass field theory for nonparametric one-dimensional variational problems and the contributions of Mayer, Kneser, Hilbert and Carathéodory. The corresponding field theory for parametric integrals is presented in Chapter 8. There we have also a first glimpse at the so-called *direct method* of the calculus of variations. This is a way to establish directly the existence of minimizers by means of set-theoretic arguments; another treatise will entirely be devoted to this subject. In addition we sketch field theories for multiple integrals at the end of Chapters 6 and 7.

In chapter 7 we describe an important involutory transformation, which will be used to derive a dual picture of the Euler–Lagrange formalism and of field theory, called *canonical formalism*. In this description the dualism *ray versus wave* (or: particle–wave) becomes particularly transparent. The canonical formalism is a part of the *Hamilton–Jacobi theory*, of which we give a self-contained presentation in Chapter 9, together with a brief introduction to symplectic geometry. This theory has its roots in Hamilton's investigations on geometrical optics, in particular on systems of rays. Later Hamilton realized that his formalism is also suited to describe systems of point mechanics, and Jacobi developed this formalism further to an effective integration theory of ordinary and partial differential equations and to a theory of canonical mappings. The connection between *canonical* (or *symplectic*) *transformations* and *Lie's theory of contact transformations* is discussed in Chapter 10 where we also investigate the relations between the principles of Fermat and Huygens. Moreover we treat *Cauchy's method* of integrating partial differential equations of first order by the method of characteristics and illustrate the connection of this technique with Lie's theory.

The reader can use the detailed table of contents with its numerous catchwords as a guideline through the book; the detailed introductions preceding each chapter and also every section and subsection are meant to assist the reader in obtaining a quick orientation. A comprehensive *glimpse at the literature* on the Calculus of Variations is given at the end of Volume 2. Further references can be found in the Scholia to each chapter and in our bibliography. Moreover, important historical references are often contained in footnotes. As important examples are sometimes spread over several sections, we have added a *list of examples*, which the reader can also use to locate specific examples for which he is looking.