Phys374, Spring 2008, Prof. Ted Jacobson
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Laplacian

The Laplacian of a scalar function f is the divergence of the curl of f,

$$\nabla^2 f = \nabla \cdot \nabla f = \partial_x^2 f + \partial_y^2 f + \partial_z^2 f, \tag{1}$$

where the last expression is given in Cartesian coordinates, and $\partial_x^2 f$ means $\partial^2 f/\partial x^2$, etc. The textbook shows the form in cylindrical and spherical coordinates.

There is a nice formula that is not in the textbook, and I have rarely if ever seen it written, but I think it's sweet and useful so I record it here:

$$\int_{\mathcal{V}_r} \nabla^2 f \, dV = 4\pi r^2 \frac{d}{dr} \left[f_{\mathrm{avg}}(\partial \mathcal{V}_r) \right]. \tag{2}$$

The integral on the left hand side is over a spherical ball $\mathcal{V}_r$ of radius r, and the notation on the right denotes the average of f over the spherical boundary $\partial\mathcal{V}_r$ of this ball. Here's the proof:

$$\begin{aligned}
\int_{\mathcal{V}_r} \nabla \cdot \nabla f \, dV &= \int_{\partial\mathcal{V}_r} \nabla f \cdot \mathbf{dS} && (3)\\
&= \int_{\partial\mathcal{V}_r} \nabla f \cdot \hat{\mathbf{r}} \, r^2 d\Omega && (4)\\
&= r^2 \int_{\partial\mathcal{V}_r} \partial_r f \, d\Omega && (5)\\
&= 4\pi r^2 \frac{d}{dr} \int_{\partial\mathcal{V}_r} f \, d\Omega/4\pi. && (6)
\end{aligned}$$

Here $d\Omega = \sin\theta d\theta d\varphi$ is the solid angle element on the sphere, whose integral over the whole sphere is equal to 4π. The integral on the right hand side is therefore just the average of f over the surface of the sphere.

The same idea works in other dimensions. For example in 2d, the ball becomes a disk with circular boundary, $4\pi r^2$ is replaced by the circumference of the circle $2\pi r$, and the $\delta\Omega/4\pi$ is replaced by $d\varphi/2\pi$. It even words in 1d: then the ball becomes an interval $[x-r, x+r]$ with boundary consisting of the two endpoints $\{x-r, x+r\}$. Then the result becomes $\int_{x-r}^{x+r} d^2 f/dx^2 \, dx = 2(d/dr)[(f(x+r) + f(x-r))/2]$, twice the rate of change of the average of the two endpoints with respect to the "radius" r. **Exercise**: Verify this 1d result directly using the fundamental theorem of calculus.

Laplace's equation

Laplace's equation $\nabla^2 f = 0$ comes up a lot. A function satisfying this equation is called *harmonic*. The previous result shows that the average of a harmonic function over a spherical surface is equal to the average over any other spherical surface concentric with the first one, including the infinitesimal sphere at the middle. Therefore,

> *the average of a harmonic function over the surface of any sphere is equal to the value of the function at the center of the sphere.*

The same thing holds for harmonic functions in 2d, with the spherical surface replaced by a circle. The 1d analog of this result is easy to understand: if $d^2 f/dx^2 = 0$, then $f(x)$ is a linear function, so its value at the center of any interval is halfway between its values at the two endpoints, so it is equal to the average of these two values.

Maxima and minima

At a local extremum of a function of one variable $f(x)$, the first derivative vanishes. The second derivative is positive at a minimum and negative at a maximum. At a minimum of a function of three variables $f(x, y, z)$, the function is minimum with respect to variations of any of the three variables, so $\partial_x^2 f$, $\partial_x^y f$, and $\partial_z^2 f$ are all positive. Their sum is therefore positive, so $\nabla^2 f > 0$. Similarly at a maximum it must be that $\nabla^2 f < 0$. A harmonic function thus cannot have any local maxima or minima, since it has $\nabla^2 f = 0$ everywhere. This can also be seen from the "average value theorem": if f has a local maximum at a point $\mathbf{x}_0$, then $f(\mathbf{x}_0)$ is greater than the value of f at all nearby points, so it must be greater than the average value over a small sphere centered on $\mathbf{x}_0$. Therefore it must not satisfy $\nabla^2 f = 0$ at $\mathbf{x}_0$.