HW#9 —Phys374—Spring 2008
Due before class, Friday, April 18, 2008
www.physics.umd.edu/grt/taj/374c/

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Note: *Be sure to fully justify the relation between the given integral and the closed contour integral you employ in the following problems.*

1. Find the residues of the following functions at the given values of z:

   (a) $(z+z^2)^{-1}$ at 0 and at -1.
   (b) $z^{-2}\ln(1+2z)$ at 0
   (c) $[z^3(z+2)^2]^{-1}$ at 0 and at -2.
   (d) $\cos z/(2z-\pi)^4$ at $\pi/2$
   (e) $(z^2+1)^{-3}$ at $\pm i$.

   Hint: See the supplement for methods of evaluating residues. [$3\times 5 = 15$ pts.]

2. Consider the real integral
$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$$
   where a and b are positive real numbers.

   (a) Evaluate the integral using contour integration assuming $a \neq b$ (so there are only simple poles).
   (b) Evaluate the integral using contour integration assuming $a = b$ from the beginning (so the poles are of order 2), and then check that you recover the same result by setting $a = b$ in the result of the previous part.

   Hint: You can check your result by testing for a few properties: the integral is manifestly positive and symmetric under interchange of a and b, and scales as λ^{-3} under the scaling $a \to \lambda a$ and $b \to \lambda b$. (Equivalently, if you think of a and b as having dimensions, the integral has the same dimensions as a^{-3}.) [5+5=10 pts.]

3. (a) Show using contour integration that
$$\int_0^{\infty} \frac{\cos mx\, dx}{x^2+a^2} = \frac{\pi}{2a}e^{-ma}$$
   Hint: See the similar example in the textbook.

   (b) Explain in words why the result decays so rapidly (i) as m grows with a fixed, and (ii) as a grows with m fixed. [8+2=10 pts.]

4. (a) Evaluate the integral $\int_0^\infty dx/(x^n+1)$, where $n \geq 2$ is a positive integer, by relating it to the contour integral around the boundary of an infinite piece of pie with edges $\theta = 0$ and $\theta = 2\pi/n$, together with the arc at infinity that joins these edges. (b) Show that the result approaches 1 as $n \to \infty$, and explain with reference to the behavior of the integrand why this is the limiting value. (*Answer*: $(\pi/n)/\sin(\pi/n)$.) [8+2=10 pts.]

5. The relation between the real Fourier coefficients for the sine and cosine terms can be obtained with the help of the following identities:

$$\int_{-\pi}^{\pi} \cos(m\theta)\, \cos(n\theta)\, d\theta \;=\; \pi\delta_{mn} \tag{1}$$

$$\int_{-\pi}^{\pi} \sin(m\theta)\, \sin(n\theta)\, d\theta \;=\; \pi\delta_{mn} \tag{2}$$

$$\int_{-\pi}^{\pi} \cos(m\theta)\, \sin(n\theta)\, d\theta \;=\; 0, \tag{3}$$

where m and n are assumed to be positive integers. (These are equivalent to eqns (15.3-6) in the textbook.) Prove these identities by expressing the cosine and sine in terms of complex exponentials, and using $\int_{-\pi}^{\pi} e^{ik\theta}\, d\theta = 2\pi\delta_{k0}$. (Here δ_{kl} is the Kronecker delta, equal to 1 if the integers k and l are equal, and zero otherwise).
[5 pts.]