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1. (a) Evaluate $\int z\,dz$ along the following two contours connecting -1 to 1: (i) from -1 to 1 along the real axis, and (ii) along the semicircle of radius 1 in the complex plane. (Do both contour integrals explicitly. Use the polar angle as the variable of integration for (ii), and use $x = Re(z)$ as the variable of integration for (i).) Explain how you could have known the two integrals would agree without even evaluating them. (b) Repeat part (a) for the integral $\int z^*\,dz$, and explain why the two contour integrals do *not* agree in this case. [7+3=10 pts.]

2. **Fluid flow and analytic functions**: Problem 16.3 c,d,e,f [2+3+3+2=10 pts.]

3. **Vortex**: The velocity potential for a point source of fluid flow is given by the real part of $h(z) = k \ln z$ (where k is a constant), as shown in the previous problem. Show that the *imaginary part* of $h(z)$ is the velocity potential for a point vortex. Do this by showing that the flow lines are circles centered on $z = 0$, and find the relation between k and the circulation. [5 pts.]

4. (a) Show that a flow with complex velocity potential $h(z)$ has speed $|dh/dz|$. (*Hint*: Use the fact that $dh/dz = \partial_x h$, and use the Cauchy-Riemann equations. (b) Apply this to the previous two problems to find the flow speed as a function of r.
[8+2=10 pts.]

5. Consider potential flow in the wedge $0 < \theta < \alpha$ of the plane, bounded by walls at $\theta = 0$ (the x axis) and $\theta = \alpha$. At the walls the velocity must be parallel to the walls. (a) Show that the velocity potential $h(z) = Az^{\pi/\alpha}$ satisfies this boundary condition. (b) Sketch the flow lines for this potential for the cases $\alpha = \pi/2$, $\alpha = \pi$, $\alpha = 3\pi/2$, and $\alpha = 2\pi$. (c) Find the speed of the flow as a function of position on the plane for general α. Explain why your result makes sense for the case $\alpha = \pi$. (d) If $\alpha < \pi$, the speed goes to infinity as r goes to infinity. Explain how this is compatible with incompressibility of the flow. [3+3+2+2=10 pts.]

   Note that if $\alpha < \pi$ the speed goes to zero at the vertex of the wedge, whereas if $\alpha > \pi$ the speed goes to infinity at the vertex. I read that this is why the wind whistles when going over a pointed obstacle: locally supersonic velocities are reached.

   (*one more problem ...*)

6. Consider potential flow perpendicular to an infinite solid cylinder of radius R. This reduces to a two-dimensional problem in the xy plane. The cylinder intersects the plane in a disk. For the boundary condition "at infinity", suppose that far from the cylinder in all directions the velocity is $\mathbf{v} = v_0\hat{\mathbf{x}}$. The boundary condition at the cylinder surface is that there is no flow perpendicular to the cylinder, so $\mathbf{v} \cdot \hat{\mathbf{r}} = 0$, taking the origin of polar coordinates at the center of the cylinder. That is, the partial derivative of the velocity potential with respect to radius r (at fixed angle) vanishes at $r = R$. To solve for the flow we need only find an analytic function of $z = x + iy$ whose real part satisfies the appropriate boundary conditions. [2+5+3+2+2+1=15 pts.]

(a) Find an analytic function $h_1(z)$ whose real part is a potential for the velocity field $\mathbf{v} = v_0\hat{\mathbf{x}}$.

(b) Find an analytic function $h_2(z)$, to be added to your function $h_1(z)$ from the previous part, such that the real part of $h(z) = h_1(z) + h_2(z)$ satisfies the boundary condition everywhere on the cylinder, as well as the boundary condition at infinity. To do this assume $h_2(z) = az^n$, and find the values for constants a and n for which the boundary conditions are satisfied everywhere on the cylinder. (*Hint*: Use polar coordinates. The result, which you are supposed to derive, is $h(z) = v_0(z + R^2/z)$.)

(c) Using your result from the previous part, find the velocity (magnitude and direction) at the point $(x, y) = (0, R)$ on the surface of the cylinder. How does it compare with v_0?

(d) Find the equation for the flow line that goes through the point $(x, y) = (0, y_0)$ (with $y_0 > R$). (The equation should involve x, y, y_0, v_0 and R.)

(e) Find the asymptotic y value y_∞ when $x \to \infty$ for the flow line of problem 6d.

(f) Sketch the flow lines of problem 6d for the cases $y_0 = R$ and $y_0 = 2R$, along with the circle of radius R representing the cross-section of the cylinder.