HW#4 —Phys374—Spring 2008 | Prof. Ted Jacobson
Due before class, Friday, Feb. 29, 2008 | Room 4115, (301)405-6020
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1. Problems 5.2a,b. (*Pressure gradient*) (For 5.2a, scan, photocopy, or trace the map.) For 5.2b, you'll have to work out or look up the distance across Ireland in the relevant direction.) [5+5=10 pts.]

2. Consider the function $f(x, y, z) = ax^2 + by^2 + cz^2$. [2+2+2+2+2=10 pts.]

   (a) Find ∇f.

   (b) Find the rate of change of f at the point $(1, 1, 1)$ in the direction of the position vector $\mathbf{r}$. (*Caution*: $\mathbf{r}$ is not a unit vector.)

   (c) Find the rate of change of f at the point $(1, 1, 1)$ in the direction of most rapid increase of f.

   (d) The level sets $f =$ const. are ellipsoids. Find the unit normal to the ellipsoid at the point $(1, 1, 1)$.

   (e) What is the angle between $\mathbf{r}$ and the normal to the ellipsoid at $(1, 1, 1)$? Check that in the spherically symmetric case $a = b = c$ the angle is zero.

3. Derive the following identities [2+2=4 pts]:

   (a) If f is a scalar field and $\mathbf{v}$ is a vector field then
   $$\nabla \cdot (f\mathbf{v}) = \nabla f \cdot \mathbf{v} + f\nabla \cdot \mathbf{v}. \tag{1}$$

   (b) If f is a scalar field and h is a function of one variable, then
   $$\nabla h(f) = h'(f)\nabla f. \tag{2}$$

4. In this problem r and $\mathbf{r}$ are the distance and the position vector from the origin. [20 pts.]

   (a) (i) Show using both cartesian and spherical coordinates that $\nabla r = \hat{\mathbf{r}}$. (ii) Explain why this is dimensionally balanced. (iii) Derive this equation by a geometrical discussion of the properties of the direction and magnitude of ∇r. [(2+2)+2+2=8 pts.]

   (b) Show that $\nabla \cdot \mathbf{r} = 3$. [2 pts.]

   (c) Show that $\nabla \cdot \hat{\mathbf{r}} = 2/r$ by the following method: write $\hat{\mathbf{r}} = r^{-1}\mathbf{r}$, and use the results of problem 3 and problem 4b. [2 pts.]

   (d) Show that if $\mathbf{m}$ is a constant vector, then (i) $\nabla(\mathbf{m} \cdot \mathbf{r}) = \mathbf{m}$ and (ii) $\nabla(\mathbf{m} \cdot \hat{\mathbf{r}}) = r^{-1}(\mathbf{m} - (\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}})$. [2+2=4 pts.]

   (e) The magnetic field of a dipole moment $\mathbf{m}$ is $\mathbf{B}(\mathbf{r}) = (3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m})/r^3$. Use the above results to show that this has zero divergence (as does *any* magnetic field satisfying Maxwell's equations). (The textbook does this using the explicit cartesian components of $\mathbf{B}$ for the case $\mathbf{m} = m\hat{\mathbf{z}}$.) [4 pts.]

(*more problems follow*)

5. Problems 8.2a,b,c,d (*Gravitational field of a spherically symmetric mass*) For part (c) you may just sketch the graph. No careful plot is required. [2+2+4+2=10 pts.]

6. Newton's law for the gravitational acceleration vector $\mathbf{g}$ produced by a source mass density ρ is $\nabla \cdot \mathbf{g} = -4\pi G\rho$. In the presence of a "cosmological constant" Λ (the greek letter "Lambda"), this is modified to read

$$\nabla \cdot \mathbf{g} = -4\pi G\rho + \Lambda c^2, \tag{3}$$

where c is the speed of light. [2+2+1+3+2=10 pts.]

   (a) What are the dimensions of Λ?

   (b) Assuming a spherically symmetric, source mass M and a spherically symmetric field, use (3) and Gauss' theorem to determine the gravitational acceleration vector field $\mathbf{g}(\mathbf{r})$ at any point outside the mass. Notice that the cosmological constant term produces a repulsive contribution to the force.

   (c) Suppose $\rho = 0$ everywhere but $\Lambda \neq 0$. Choose an arbitrary center of spherical symmetry, and write the spherically symmetric solution for $\mathbf{g}(\mathbf{r})$.

   (d) Show that the test particle equation of motion $d^2\mathbf{r}/dt^2 = \mathbf{g}(\mathbf{r})$ in this field implies that a particle accelerates away from the origin with an acceleration proportional to its distance from the origin. Show that the equation has solutions that expand exponentially as $\mathbf{r}(t) = e^{\alpha t}\mathbf{r}_0$, with α and $\mathbf{r}_0$ constant. What is α?

   (e) Consider two test particle trajectories, $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ in this field. Show that the separation vector $\mathbf{s} = \mathbf{r}_2 - \mathbf{r}_1$ satisfies exactly the same equation as they each satisfy. This means that any two particles accelerate away *from each other* with an acceleration proportional to their separation distance.

   This is a Newtonian model of the exponential expansion of the universe due to a cosmological constant. The relative acceleration is independent of the arbitrary origin that was chosen in the original solution for $\mathbf{g}(\mathbf{r})$. *A change of the origin would just add a constant vector to* $\mathbf{g}(\mathbf{r})$, *shifting the acceleration of all particles together, and not affecting their relative acceleration.*