HW#10 —Phys374—Spring 2008
Due before class, Friday, April 25, 2008
www.physics.umd.edu/grt/taj/374c/

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1. Consider the "rectified cosine function" defined by

$$f(x) = \cos(\pi x/2L), \qquad L \leq x \leq L, \tag{1}$$

and continued periodically so that $f(x + 2L) = f(x)$. [2+3+5+5=15 pts.]

   (a) Sketch the function $f(x)$ over several periods.

   (b) Use the symmetry to explain why the Fourier coefficients b_n vanish.

   (c) Find the non-vanishing Fourier coefficients. (*Hints*: (i) To clean things up, change variables to $\theta = \pi x/L$. (ii) You'll need to do a probably unfamiliar integral, which you can look up or work out for yourself.)

   (d) Using a computer program (Mathematica, Maple, Matlab, or something else) plot the sum of the first few terms in the Fourier series, together with (1), for $\theta \in (-2\pi, 2\pi)$. Show the result with 1 (just the constant part), 2, 5, and 20 terms included. With 5 terms the sum should already be quite close to (1), except near the zeros where the slope is discontinuous.

2. In section 11.5, *Explosion of a nuclear bomb*, and hw6, the neutron density is assumed to have a factored form $N(r, t) = F(r)H(t)$, and we found the equations satisfied by $F(r)$ and $H(t)$. Then we wrote $F(r) = f(r)/r$ and found that $f(r)$ must be a sin function. After applying the boundary conditions $f(0) = 0 = f(R)$ the solution took the form

$$N_n(r, t) = A_n \exp(\mu_n t) \sin(k_n r)/r, \tag{2}$$

where n is a positive integer, A_n is an arbitrary constant, $k_n = n\pi/R$, and μ_n is determined by the diffusion constant κ, the production rate λ, the radius of the sphere R and the integer n. A general solution is a linear combination of such solutions, $N(r, t) = \sum_n N_n(r, t)$, with different values of the constants A_n.

Once the coefficients A_n are known, $N(r, t)$ is determined for all time. Consider for example the case when initially at $t = 0$ there is a constant density of neutrons N_i in a sphere of radius $a < R$, and no neutrons outside that sphere. Find the values of the coefficients A_n in this case, and use these to write out the function $N(r, t)$ as an explicit series. [5 pts.]

(*Hint*: To evaluate the coefficients A_n, I suggest you multiply $N(r, 0)$ by $r \sin(k_n r)$ and integrate over r from 0 to R. Using the given initial density you'll get one value, and using the series expansion you'll encounter integrals very close to (15.3,5) in the textbook, with L replaced by R and with the range of integration cut in half. The latter will be proportional to A_n, so you'll be able to solve for A_n.)

3. Find the Fourier transform of $f(t) = A\sin(\omega_0 t + \varphi)$. [10 pts.]

4. Problems 15.6 g,h (Fourier transform of correlation and Parseval's theorem) [10 pts.] (*Note*: The conventions (15.42), (15.43) are used here.)

5. **Sampling Theorem**

> *Exact reconstruction of a continuous-time signal from its discrete-time samples is possible if the signal is band-limited and the sampling frequency is greater than twice the signal bandwidth.*

Consider a signal $f(t)$ whose Fourier transform $F(\omega)$ is zero for $|\omega| > \Omega$,

$$f(t) = \int_{-\Omega}^{\Omega} F(\omega)\, e^{-i\omega t}\, d\omega. \tag{3}$$

This is called a *band-limited* signal. (Note the exponent sign convention of (15.42) is used here. See section 15.5 for a discussion of the alternate conventions.) Evaluating (3) at the discrete times $t = nt_s$, where the *sampling time* t_s is defined by $t_s = \pi/\Omega$, yields

$$f(nt_s) = \int_{-\Omega}^{\Omega} F(\omega)\, e^{-in\pi\omega/\Omega}\, d\omega. \tag{4}$$

The right hand side of (4) is recognized as 2Ω times the nth coefficient in the Fourier series for $F(\omega)$ on the interval $(-\Omega, \Omega)$. Being limited to this interval, the function $F(\omega)$ is determined by these Fourier coefficients, and therefore by the discrete "samples" $f(nt_s)$. The sampling frequency $1/t_s = \Omega/\pi$ is twice the bandwidth $\Omega/2\pi$.

Show that $f(t)$ can be reconstructed explicitly from the samples $f(nt_s)$ via

$$f(t) = \sum_{n=-\infty}^{\infty} f(nt_s)\, \frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi}. \tag{5}$$

[10 pts.]
(*Hint*: Write $F(\omega)$ as a Fourier series, substitute in (3), and integrate over ω.)