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Gravity waves on water

Waves on the surface of water can arise from the restoring force of gravity or of surface tension, or a combination. For wavelengths longer than a couple of centimeters surface tension can be neglected, and the waves are called *gravity waves*. Short wavelength surface waves are called *capillary waves*. Dimensional analysis told us that the speed of gravity waves with wavelength λ much shorter than the depth of the body of water but still long enough to ignore surface tension must be proportional to $\sqrt{g\lambda}$, while those with wavelength much longer than the depth have speed proportional to $\sqrt{gh}$. In this supplement we'll derive the speed for the general case. Actually we'll do more: we'll find the motion of the water, as a function of time and depth, for small amplitude waves.

Incompressible flow

Mass conservation is expressed by the continuity equation, $\partial_t\rho+\nabla\cdot(\rho\mathbf{v}) = 0$. If, like water (but unlike, say, a gas), the fluid hardly compresses, then the mass density ρ is nearly constant in space and time, so the continuity equation reduces to $\nabla\cdot\mathbf{v} = 0$. By Gauss's theorem this implies the vanishing of the fux integral $\int_{\partial\mathcal{V}} \mathbf{v}\cdot\mathbf{dS}$ for any volume $\mathcal{V}$. That is, the net volume flow into or out of any region is zero. Equivalently, the flow is volume-preserving.

Fluid equation of motion

The mass and volume of a portion of fluid is constant as it is carried along in an incompressible flow. Newton's law $\mathbf{F} = m\mathbf{a}$ thus applies to each infinitesimal "fluid element" of volume δV, which has a fixed mass $\delta m = \rho\delta V$.[1] The acceleration of a fluid element with trajectory $\mathbf{r}(t)$ is the rate of change of its velocity $\mathbf{v}(\mathbf{r}(t), t)$ as it is carried along by the flow. This receives contributions both from any explicit time dependence in $\mathbf{v}$ and from the fact that the flow may carry it into a location where the velocity field is different. That is, the acceleration is the *convective derivative* of the velocity,

$$\mathbf{a} = \partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}. \tag{1}$$

[1]For compressible flows one must use the more general form of Newton's second law, $\mathbf{F} = d\mathbf{p}/dt$, where $\mathbf{p}$ is the momentum of a fluid element, which can change both because of acceleration and because of mass flow. See §11.3,6 of the textbook for details.

(See section 5.5 of the textbook for a discussion of the convective derivative.)

The force on a fluid element arises from internal forces of pressure p and viscosity μ, as well as external forces such as gravity. The pressure force is $-\nabla p\,\delta V$ (cf. section 5.2 of the textbook), and the viscous force is $\mu\nabla^2\mathbf{v}\,\delta V$ (cf. section 11.6 of the textbook, although I'm not convinced that the derivation there is really valid). The gravitational force is $-\delta m\,\nabla\Phi$, where Φ is the gravitational potential per unit mass.

The acceleration of a fluid element is equal to the total force divided by the mass, hence

$$\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\rho^{-1}\nabla p + (\mu/\rho)\nabla^2\mathbf{v} - \nabla\Phi. \tag{2}$$

This is a special case of the *Navier-Stokes equation*, for incompressible flow. With the viscosity term neglected it's called the *Euler equation*.

Irrotational flow

The vorticity of the flow is defined as $\boldsymbol{\omega} = \nabla\times\mathbf{v}$. A remarkable fact is that, for incompressible flow with conservative external forces like gravity, only boundary effects can generate vorticity if no vorticity exists initially. To see why, just take the curl of (2). Since ρ is constant the pressure force is a gradient, as is the gravitational force, so the curls of both of these vanish. The curl of the viscous force is $(\mu/\rho)\nabla^2\boldsymbol{\omega}$. The curl of the first term on the left hand side is just $\partial_t\boldsymbol{\omega}$. For the second term we need an identity:

$$\nabla\times\Big((\mathbf{v}\cdot\nabla)\mathbf{v}\Big) = (\mathbf{v}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{v}, \tag{3}$$

where the incompressibility condition $\nabla\cdot\mathbf{v} = 0$ has been used. (This identity is not hard to prove using an index notation.) Thus we've found the evolution law for the vorticity,

$$\partial_t\boldsymbol{\omega} + (\mathbf{v}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{v} + (\mu/\rho)\nabla^2\boldsymbol{\omega}. \tag{4}$$

If the vorticity is initially zero, then according to this equation its time derivative is zero, so it remains zero!

Vorticity can be generated by boundary effects such as the no-slip condition at a surface, or the appearance of capillary waves, or the action of non-conservative forces, or violation of the incompressibility condition. All of these effects can be neglected for small amplitude gravity waves, so we may assume the vorticity is zero in the wave, i.e. the flow is *irrotational*.

Exercise a: (i) Show that in the presence of a boundary with the no slip condition $\mathbf{v} = 0$ at the boundary, the vorticity cannot be zero. (ii) Explain mathematically in terms of the above derivation why a non-conservative force can generate vorticity.

Small amplitude assumption

The term $(\mathbf{v} \cdot \nabla)\mathbf{v}$ in (2) called the *advective* term (cf. section 12.3 of the textbook). Since this term is quadratic in $\mathbf{v}$, one might expect that for small velocities it is negligible, but what exactly does "small" mean? A velocity has dimensions LT^{-1}, so it is meaningless to say it is "small" in absolute terms. The real question is whether it is small compared to the other terms in the equation.

To compare the size of $(\mathbf{v} \cdot \nabla)\mathbf{v}$ with $\partial_t \mathbf{v}$, suppose we are considering a wave with amplitude v_0, wavelength λ and period T. Then a spatial derivative scales as $1/\lambda$ and a time derivative scales as $1/T$. (See Section 12.3 of the textbook for further discussion.) Then

$$(\mathbf{v} \cdot \nabla)\mathbf{v} \sim \frac{v_0^2}{\lambda} \qquad \text{and} \qquad \partial_t \mathbf{v} \sim \frac{v_0}{T}, \tag{5}$$

so

$$\frac{\text{advective term}}{\text{time derivative}} \sim \frac{v_0 T}{\lambda} = \frac{v_0}{c}, \tag{6}$$

where $c = \lambda/T$ is the wave velocity. The advective term is therefore negligible provided the fluid velocity is much smaller than the wave speed.

Since we don't yet know the wave speed this is not easy to assess, so lets back up one step in (6). The numerator $v_0 T$ is of order the maximum displacement of a fluid element. That is, it's the amplitude of the wave motion. If this is much smaller than the wavelength, then the advective term can be neglected. Let's make this assumption from now on. This assumption *linearizes* the equation of motion and makes the problem *much* easier to solve.

Velocity potential

Since the velocity has zero curl for irrotational flow it must be the gradient of a scalar, $\mathbf{v} = \nabla f$. The function f is called the *velocity potential*, and the flow is also called *potential flow*. If an irrotational flow is also incompressible, then $\nabla^2 f = 0$, i.e. its velocity potential satisfies Laplace's equation. This represents a huge simplification. We have traded the vector $\mathbf{v}$ for the scalar φ, which satisfies a simple *linear* equation!

But the situation isn't as simple as it appears, since we still must implement the Navier-Stokes equation (2) which determines the time dependence of φ. This task will be greatly eased by approximations valid for small amplitude motion with sufficiently long wavelenths: neglect of the non-linear term and the viscosity term.

Under the small amplitude assumption, all significant terms in the fluid equation (2) are gradients, and the equation can be expressed as

$$\nabla(\partial_t f + p/\rho + \Phi) = 0. \tag{7}$$

Hence under these assumptions the quantity $\partial_t f + p/\rho + \Phi$ is constant in space, that is,

$$\partial_t f + p/\rho + \Phi = F(t), \tag{8}$$

where $F(t)$ is some undetermined function depending on time but not on space. The undetermined function $F(t)$ can be absorbed into $f(\mathbf{r}, t)$ by the spatially constant shift $f \to f - \int^t F(t)\, dt$, which does not affect the velocity $\mathbf{v} = \nabla f$. The fluid equation for small amplitude motion thus becomes simply

$$\partial_t f + p/\rho + \Phi = 0, \tag{9}$$

together with Laplace's equation $\nabla^2 f = 0$.

The relevant solution to Laplace's equation for our problem will be determined by imposing the boundary conditions. These will be that (i) at the bottom, the fluid has no vertical component of velocity, and (ii) at the top surface, the pressure is everywhere equal to a fixed, atmospheric pressure.

Fixing a wavelength

The first step in deriving the form of a gravity wave with wavelength λ is to assume the velocity potential $f(\mathbf{r}, t)$ has a factorized form with this wavelength,

$$f(\mathbf{r}, t) = A(t)G(y)\sin kx \tag{10}$$

where the wave number is $k = 2\pi/\lambda$. Here we have assumed that $\hat{\mathbf{x}}$ is the wave propagation direction, $\hat{\mathbf{y}}$ is the vertical direction, and the wave is completely independent of the remaining direction $\hat{\mathbf{z}}$. This means the wave crests will be infinitely long straight lines in the z-direction. We choose the surface of the undisturbed fluid to lie at $y = 0$. The general solution will be a linear combination of solutions of this factorized form, though we won't prove that here.

Imposing the incompressibility condition

Next we impose the condition that f satisfies Laplace's equation. Since we fixed the wavelength and assumed no z-dependence, this yields

$$\begin{aligned} \nabla^2 f &= (\partial_x^2 + \partial_y^2 + \partial_z^2) f && (11) \\ &= (-k^2 + G''/G) f && (12) \end{aligned}$$

where G'' stands for d^2G/dy^2. Laplace's equation $\nabla^2 f = 0$ thus reduces to a simple ordinary differential equation for $G(y)$,

$$G'' = k^2 G. \tag{13}$$

The general solution to this equation is

$$G(y) = c_1 e^{ky} + c_2 e^{-ky}. \tag{14}$$

Boundary condition at the bottom

If the container is infinitely deep the "bottom" boundary condition is simplest: $G(y \to -\infty) = 0$. Using (14) this implies $c_2 = 0$. We should not try to determine the other constant c_1, since it represents just the overall scale of the wave amplitude, which is free.

If the container has a finite depth h, then the boundary condition to be imposed at $y = -h$ is that the vertical component of velocity vanishes there (no flow through the bottom). To impose this, remember that $\mathbf{v} = \nabla f$, so $v_y = 0 \iff \partial_y f = 0 \iff G' = 0$. Thus our boundary condition at the bottom is $G'(-h) = 0$. Since

$$G'(y) = c_1 k e^{ky} - c_2 k e^{-ky} \tag{15}$$

we infer $c_2 = e^{-2kh} c_1$. When $h \to \infty$ this yields $c_2 = 0$, consistent with what we got previously. For a general h we have

$$\begin{aligned} G(y) &= c_1 (e^{ky} + e^{-2kh} e^{-ky}) && (16) \\ &= c_1 e^{-kh} (e^{k(y+h)} + e^{-k(y+h)}) && (17) \\ &= 2c_1 e^{-kh} \cosh\Big(k(y+h)\Big). && (18) \end{aligned}$$

At the bottom ($y = -h$) the cosh equals 1, while at the top it's $\cosh kh$, so G grows as y goes from bottom to top values. The constant coefficient $2c_1 e^{-kh}$ can be absorbed into the definition of $A(t)$ in (10).

Boundary condition at the top

What can be the boundary condition at the top? What is fixed there? The surface is moving up and down, but the pressure there is always the same and equal to the ambient atmospheric pressure $p_{\rm atm}$ above the surface. The constant $p_{\rm atm}/\rho$ can be absorbed into f by the shift $f \to f + (p_{\rm atm}/\rho)t$, since this has no effect on $\mathbf{v} = \nabla f$. *On the surface* (8) then reads

$$(\partial_t f + \Phi)\Big|_{y=y_s} = 0, \tag{19}$$

where $y_s(x,t)$ is the height of the surface at position x and time t.

The gravitational potential at the surface is $\Phi = gy_s$ (up to an arbitrary constant). The time derivative $\partial_t y_s$ is the y-component of the fluid velocity at the surface, which is related to the velocity potential by $\partial_t y_s = \partial_y f$. Thus, the time derivative of (19) yields an equation involving only f,

$$(\partial_t^2 f + g\,\partial_y f)\Big|_{y=y_s} = 0. \tag{20}$$

This is our boundary condition at the top. It will now be used to determine the time dependence of $A(t)$.

Referring to the factorized form of f (10) we have

$$\begin{aligned} \partial_t^2 f &= (\ddot{A}/A) f && (21)\\ \partial_y f &= (G'/G) f, && (22) \end{aligned}$$

where dot stands for ∂_t and prime stands for ∂_y. Inserting these into (20), and dividing by the common factor of f, we obtain

$$\ddot{A}/A = -g\,G'/G, \tag{23}$$

with the right hand side being evaluated at $y = y_s$. Using the result (18) for $G(y)$ then yields

$$\ddot{A}/A = -gk\tanh\Big(k(y_s + h)\Big). \tag{24}$$

Small amplitude assumption revisited

Now it seems we are stuck! The left hand side of (24) is a function of only t, while the right hand side is a function of $y_s(x,t)$. This means that there is no solution. How could we reach such an impasse? Recall that we have already made the assumption that the amplitude $y_{s,\max}$ is much smaller than the wavelength, so $ky_s \ll 1$. Thus, provided the amplitude is also much smaller than the depth h, it is a good approximation to replace $\tanh(k(y_s + h))$ by the *constant* expression $\tanh(kh)$.

If the amplitude is *not* much smaller than the depth, then we'd have to keep the y_s term, with which there is no solution to (24) due to the x dependence of $y_s(x,t)$. (Also the t dependence would imply $A(t)$ is not sinusoidal.) The resolution of this puzzle is that in fact we already assumed not only that the wave amplitude is small compared to the wavelength, but rather that the advective term in the acceleration is must smaller than the partial derivative with respect to time, $(\mathbf{v} \cdot \nabla)\mathbf{v} \ll \partial_t \mathbf{v}$. Consideration of the derivative in the x direction brings in the wavelength, but we must also consider the derivative in the y-direction. Since the vertical component of the velocity vanishes at the bottom, the vertical derivative scales as $\sim v_0/h$. The same reasoning that led to the conclusion that the amplitude must be much less than the wavelength then also implies that it must be much less than the depth. Physically, if the water surface is moving up and down by an amount comparable to the depth, then one cannot neglect the part of the acceleration that comes from the water moving into a location where the flow has a different speed.

Dispersion relation for gravity waves

With the assumption $y_s \ll h$ we can safely neglect the y_s in (24), which then becomes just the harmonic oscillator equation,

$$\ddot{A} = -(gk \tanh kh)A. \tag{25}$$

The general solution is sinusoidal, and up to an arbitrary phase shift can be written $A(t) = A_0 \cos \omega t$, where A_0 is a constant amplitude, and

$$\omega = \sqrt{gk \tanh kh}. \tag{26}$$

This is the *dispersion relation* between frequency ω and wavevector k for incompressible, irrotational, "small" amplitude gravity waves in a fluid with a free surface. (See homework 2, problem 3.)

Putting together what we've found, the velocity potential (10) is given by

$$f(x,y,z,t) = A_0 \cos(\omega t)\, \sin(kx) \cosh(k(y+h)). \tag{27}$$

In the infinitely deep case, $\cosh k(y+h) \to (1/2)\exp(k(y+h))$, so the cosh should be replaced by $\exp(ky)$.

The solution (27) describes a superposition of waves propagating in either direction, since $\cos \omega t \, \sin kx = [\sin(kx - \omega t) + \sin(kx + \omega t)]/2$. By adding or subtracting the corresponding solution with the roles of sin and cos swapped we obtain unidirectional wave solutions. For example, the solution

$$f(x, y, z, t) = A_0 \cos(kx - \omega t) \cosh(k(y + h)). \tag{28}$$

describes a traveling wave in the $+x$ direction, if k and ω are both positive.

Motion of the fluid

The velocity potential f determines the velocity field $\mathbf{v} = \nabla f$, which determines the motion of the fluid. The trajectory of a fluid element is given by a time-dependent position vector $\mathbf{r}(t)$ satisfying the flow equation

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}(\mathbf{r}(t), t). \tag{29}$$

To determine one such trajectory we can start somewhere in the fluid and integrate (29). At each time, the velocity must be evaluated at the location the fluid element has flowed to at that time.

If the fluid elements don't go very far from their equilibrium locations, however, a great simplification occurs: to find the motion near some starting point $\mathbf{r}_0$ it suffices to a good approximation to just evaluate $\mathbf{v}$ at $\mathbf{r}_0$. Then finding the trajectory just requires integrating with respect to time with $\mathbf{r} = \mathbf{r}_0$ held fixed. More explicitly, let the position of the fluid element be given by

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{s}(t). \tag{30}$$

Then the flow equation (29) becomes

$$\frac{d\mathbf{s}}{dt} = \mathbf{v}(\mathbf{r}_0 + \mathbf{s}, t) \approx \mathbf{v}(\mathbf{r}_0, t). \tag{31}$$

The approximation is good when $(\mathbf{s} \cdot \nabla)\mathbf{v} \ll \mathbf{v}$. Remember now our scale analysis (5) for the advective term. The spatial derivative scales like the inverse wavelength $1/\lambda$. On the other hand $\mathbf{s}$ is the fluid displacement. So if the fluid displacements are small compared to the wavelength, the approximation will be good.

Once this approximation is made, the problem is easy: (31) then implies that the motion with $\mathbf{r}(0) = \mathbf{r}_0$ is given by

$$\mathbf{s}(t) = \mathbf{r}_0 + \int^t \mathbf{v}(\mathbf{r}_0, t) \, dt. \tag{32}$$

The lower limit is unspecified since an arbitrary small constant displacement from $\mathbf{r}_0$ can be made.

Exercise b: Consider the unidirectional wave solution (28). An applet illustrating the particle motion in the fluid can be viewed at
http://www.coastal.udel.edu/faculty/rad/linearplot.html
(i) Show that the trajectory of a fluid element near a point in the fluid with coordinates (x_0, y_0) is an elliptical motion with the same period as the wave. Choose the arbitrary integration constant in (32) so that (x_0, y_0) is the center of the ellipse. (ii) Show that the size of the ellipse is smaller and the ellipticity greater for deeper y_0. (For shallow waves the amplitude of the horizontal motion is fairly constant with depth.) (iii) Show that if the water is deep, $kh \gg 1$, the ellipses are nearly circles at the surface and remain so until y_0 is within a fraction of a wavelength of the bottom. (iv) If the wave moves to the right, is the elliptical motion clockwise or counterclockwise? Justify your answer with reference to your equations.

Note: The equation of an ellipse is $(x/a)^2 + (y/b)^2 = 1$.