HW#2 —Phys374—Spring 2007
Due before class, Friday, Feb. 9
www.physics.umd.edu/grt/taj/374b/

Prof. Ted Jacobson
Room 4115, (301)405-6020
jacobson@physics.umd.edu

1. The Newtonian gravitational potential energy of a point mass m located at radius r in the gravitational field of a spherically symmetric mass M is $\varphi(r) = -GMm/r$. For the case where M is the earth, it is natural to make a Taylor expansion of $\varphi(r)$ about the radius of the earth r_e. (a) Work out the first three terms in the Taylor series for $\varphi(r)$ about r_e in powers of $h = r - r_e$. (b) Identify the "mgh" term and give the gravitational acceleration g in terms of G, M, and r_e. (c) What is the order of magnitude of the ratio of the $O(h^2)$ term to the $O(h)$ term if h is (i) one meter, (ii) one kilometer, (iii) 350 kilometers (international space station altitude), (iv) ~ 6.6 earth radii (geosynchronous orbit)? In which cases does the $O(h)$ term give a decent approximation? [5+2+3 pts.]

2. Consider the cubic equation $ay^3 + y + 2 = 0$, with $a > 0$.

   (a) Display the location of the real roots graphically, by sketching the graphs of $y+2$ and $-ay^3$ (put y on the *horizontal* axis) and seeing where they intersect. Show in your sketch three cases: a is small, equal to, and large compared to 1. [2 pts.]

   (b) Determine the leading order a dependence of the roots in the limits (i) $a \ll 1$ and (ii) $a \gg 1$. (Don't solve it exactly, even if you can.) [8 pts.]

3. **Relativistic Energy**

   The total energy of a free particle of mass m and speed v in special relativity is

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}, \tag{1}$$

   where c is the speed of light. (i) Write out the Taylor expansion of Eq. (1) in powers of v/c up to $(v/c)^4$. (ii) Identify the physical meaning of the terms in the series. (iii) For an electron in the ground state of a hydrogen atom, $v/c \sim 0.01$. What order of magnitude would you expect for the relativistic corrections to the ground state energy compared with the non-relativistic value? [3+3+3 pts.]

4. **Speed of surface waves on shallow or deep water**

   In HW#1 you used dimensional analysis to find the speed of surface waves on water. When surface tension is irrelevant, and gravity is the only restoring force, these are called "gravity waves". The result was $v \propto \sqrt{gh}$ for shallow water ($h \ll \lambda$) and $v \propto \sqrt{g\lambda}$ for deep water ($\lambda \ll h$). It turns out that the formula interpolating between these two limits is

$$v_{ph} = \sqrt{\frac{g\tanh(kh)}{k}}, \tag{2}$$

where $k = 2\pi/\lambda$ is the wave vector. Here $\tanh(x)$ is the hyperbolic tangent defined as

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh(x)}{\cosh(x)}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}, \tag{3}$$

where $\sinh(x)$ and $\cosh(x)$ are the hyperbolic sine and cosine.

(a) Obtain the Taylor expansion of $\tanh(x)$ for small $x \ll 1$ up to the order of x^3. You can use expansions (3.14) and (3.15) from the textbook, or take derivatives of $\tanh(x)$ straightforwardly for Eq. (3.11). [3 pts.]

(b) For $x \gg 1$, show that $\tanh(x) \approx 1 - 2e^{-2x}$ in the leading and next-to-leading order. [3 pts.]

(c) Using the approximations for $\tanh(x)$ derived in parts 4a and 4b, show that Eq. (2) reduces to the forms of v quoted above in the shallow $kh \ll 1$ and deep $kh \gg 1$ limits in the leading order. [3 pts.]

(d) Using the approximations for $\tanh(x)$ derived in parts 4a and 4b, derive corrections to the forms of v quoted above in the shallow and deep limits. [3 pts.]

5. **Lagrange points L1 and L2**

If the earth's orbit around the sun were perfectly circular, there would be a point L1 inside the earth's orbit where a satellite (of negligible mass) would orbit the sun with precisely the same angular velocity as the earth, since the extra pull of the sun would be canceled by the pull of the earth (the gravitational pull of all other bodies in the solar system being neglected). There would be a similar point L2 just outside the earth's orbit where the smaller pull of the sun would be supplemented by the pull of the earth. These are called "Lagrange points".

(a) Write an equation expressing the condition that the orbital frequency at L1 is equal to that of the earth, and a similar equation for L2. Your equations should be based on Newton's law of gravity and might involve Newton's constant G, the masses of the sun and the earth m_s and m_e, the distance d_s from the sun to the earth, and the distance $d_{1,2}$ from the earth to the Lagrange points. (*Hint*: recall that centripetal acceleration on a circle can be written as $r\omega^2$, where r is the radius of the circle and ω is the orbital angular frequency.) [4 pts.]

(b) Re-write your answer to the previous part using only the dimensionless ratios $\mu = m_e/m_s$ and $\delta_{1,2} = d_{1,2}/d_s$. (*Trick*: Note that the equation for L2 is obtained from that for L1 by the substitutions $\delta_1 \to -\delta_2$ and $\mu \to -\mu$. Making these substitutions in your solution for δ_1 you can easily find δ_2.) [1 pt.]

(c) For L1, find the solution for $\delta \equiv \delta_1$ to leading order in μ, and show using the above trick that the result for L2 is identical. (One approach is to first express the equation as a fifth order polynomial equation, and then ascertain which terms give the leading order solution.) [4 pts.]

(d) Evaluate δ using the mass ratio $\mu = 1/332,830$. Then put in the (mean) solar distance $d_s \simeq 1.5 \times 10^8$ km and find $d_{1,2}$ in km. (*Answer*: $\delta_{1,2} \approx 10^{-2}$, so the distance to L1 and L2 is 1/100 times the distance to the sun, 1.5×10^6 km. This is about 4 times the distance to the moon, and about 235 times the radius of the earth.) [1 pt.]

(e) Show that the *next to leading* order term in $\delta(\mu)$ for L1 is $-3^{-5/3}\mu^{2/3}$, and the opposite sign for L2. You can do this using the "artful" method, by writing $\delta = \delta_0 + \epsilon$, where δ_0 is the leading order term you found in part 5c and ϵ is the next to leading order term, and once again ascertaining which terms give the leading order solution for ϵ. This requires a fair bit of art in this case. Alternatively, just insert an expansion of δ in powers of $\mu^{1/3}$ into the equation and equate the terms proportional to $\mu^{2/3}$. How large is this correction compared to the leading order term? [5 pts.]