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1. The relation between the real Fourier coefficients for the sine and cosine terms are obtained with the help of the following identities:

$$\int_{-\pi}^{\pi} \cos(m\theta)\,\cos(n\theta)\,d\theta = \pi\delta_{mn} \tag{1}$$

$$\int_{-\pi}^{\pi} \sin(m\theta)\,\sin(n\theta)\,d\theta = \pi\delta_{mn} \tag{2}$$

$$\int_{-\pi}^{\pi} \cos(m\theta)\,\sin(n\theta)\,d\theta = 0, \tag{3}$$

where m and n are assumed to be positive integers. Prove these identities by expressing the cosine and sine in terms of complex exponentials, and using $\int_{-\pi}^{\pi} e^{ik\theta}\,d\theta = 2\pi\delta_{k0}$. (These are equivalent to eqns (15.3-6) in the textbook.

2. Consider the "rectified cosine function" defined by

$$f(x) = \cos(\pi x/2L), \qquad L \le x \le L, \tag{4}$$

and continued periodically so that $f(x + 2L) = f(x)$.

   (a) Sketch the function $f(x)$ over several periods.

   (b) Use the symmetry to explain why the Fourier coefficients b_n vanish.

   (c) Find the non-vanishing Fourier coefficients. (*Hints*: (i) To clean things up, change variables to $\theta = \pi x/L$. (ii) You'll need to do a probably unfamiliar integral, which you can look up or work out for yourself.)

   (d) Using a computer program (Mathematica, Maple, Matlab, or something else) plot the sum of the first few terms in the Fourier series, together with (4), for $\theta \in (-3\pi, 3\pi)$. Show the result with 1 (just the constant part), 2, 5, and 50 terms included. With 5 terms the sum should already be very close to (4), except near the zeros where the slope is discontinuous.

3. Let's go back to section 11.5, *Explosion of a nuclear bomb*, and see how Fourier series can be used to evaluate the time development of a general the neutron density function $N(r, t)$. (Actually, because of the $1/r$ factor it's not quite a Fourier series, but you'll see the idea is very close.)

   In hw7, we assumed a factored form $N(r, t) = F(r)H(t)$, and found the equations satisfied by $F(r)$ and $H(t)$. Then we wrote $F(r) = f(r)/r$ and found that $f(r)$ must be a sine function. After applying the boundary conditions $f(0) = 0 = f(R)$ the solution took the form

$$N_n(r, t) = A_n \exp(\mu_n t)\sin(k_n r)/r, \tag{5}$$

where n is a positive integer, A_n is an arbitrary constant, $k_n = n\pi/R$, and μ_n is determined by the diffusion constant κ, the production rate λ, the radius of the sphere R and the integer n. A general solution is a linear combination of such solutions, $N(r,t) = \sum_n N_n(r,t)$, with different values of the constants A_n.

Consider the case when the neutron density $N(r,0)$ at time $t = 0$ is equal to $\bar{N}$ inside a sphere of radius $a < R$ and zero outside. Find the values of the coefficients A_n in this case. Given these, $N(r,t)$ is known for all time. Write out this function explicitly.

Hint: To evaluate the coefficients A_n, I suggest you multiply $N(r,0)$ by $r\sin(k_n r)$ and integrate over r from 0 to R. Using the given initial density $N(r,0)$ you'll get one value, and using the series expansion you'll encounter integrals very close to (15.5) in the textbook, with L replaced by R and with the range of integration cut in half. The latter will be proportional to A_n.