Phys374, Spring 2006, Prof. Ted Jacobson
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## Waves on a stretched string

A stretched string will vibrate when plucked. If the string is finite it oscillates back and forth, while if the string is infinite it supports traveling waves. The physical parameters defining the problem are the mass per unit length $\mu$ and tension $T$ of the string. In addition if it has finite length then the length $L$ will enter the expression for the oscillation frequency. (Don't confuse $T$ and $L$ with the same symbols used to denote generic dimensions of time and length respectively.)

**Exercise a**: Show that the combination $\sqrt{T/\mu}$ has dimensions of velocity, *and* that one cannot make a dimensionless quantity using $\mu$, $T$, and the wavelength $\lambda$. This allows us to infer that the wave speed is independent of wavelength and is proportional to this quantity.

**Exercise b**: If the string has fixed endpoints then it can vibrate at a particular set of normal mode frequencies. The lowest frequency must be proportional to some combination of the available constants $\mu$, $T$, and $L$. Find this combination.

Here we use Newton's law to derive a partial differential equation describing the motion of the string. We suppose the equilibrium configuration of the string lies along the $x$ axis, and we let $y(x,t)$ denote the perpendicular displacement of the string from its equilibrium at position $x$ and time $t$. We assume that the displacement of the string is very small, in the sense that

$$\frac{\partial y}{\partial x} \ll 1 \tag{1}$$

which means that the *slope* of the string is everywhere very small compared to one. Equivalently the *angle* $\theta$ between the string and the horizontal is small.

Since different parts of the string have different motions, we need to apply Newton's law $\mathbf{F} = m\mathbf{a}$ to each infinitesimal bit of the string separately. To this end, consider the bit of string that runs from $x$ to $x + dx$. Since we assume the slope is very small, the length of this bit of string is nearly just $dx$. The correction is of order $(dx)^2$. Neglecting this, as well as the related stretching of the string[1], we have that to lowest order in $dx$ the mass of this bit of string is

$$m = \mu dx. \tag{2}$$

The acceleration of this bit in the $y$ direction is, to zeroth order in $dx$, given by the second partial derivative with respect to $t$,

$$a_y = \frac{\partial^2 y}{\partial t^2} \tag{3}$$

evaluated at $x$.

It remains to evaluate the $y$ component of the force on this bit of string. The force arises from the vector sum of the forces due to the pull of the string on the right and on the left of the bit. The string exerts a force of magnitude equal to the tension $T$, and direction along the string. Since the direction is along the string, and since over the interval $dx$ the string is nearly straight (is exactly straight when $dx$ is infinitesimal) the $y$ component of the tension force on the left is determined by the angle $\theta$ made by the tangent to the string at $x$ and is

$$\begin{aligned} F_y^{\text{left}} &= -T \sin\theta & (4) \\ &= -T \tan\theta + O(\theta^3) & (5) \\ &= -T \frac{\partial y}{\partial x} + O\left[\left(\frac{\partial y}{\partial x}\right)^3\right]. & (6) \end{aligned}$$

The minus sign is because the string on the left pulls downward (in the negative $y$ direction) if the slope is positive. Since we assume the slope is much smaller than 1, the correction term can be neglected in what follows, but it should be clear that our result for the force is only accurate up to a correction of relative size equal to the square of the slope. The string force on the right is given by a similar expression, without the minus sign. Thus to lowest order in the slope the net force in the $y$ direction is

$$F_y = T \frac{\partial y}{\partial x}\Big|_{x+dx} - T \frac{\partial y}{\partial x}\Big|_{x}. \tag{7}$$

[1] The string mass is fixed. When the string is displaced from equilibrium and is therefore lengthened its mass per unit length must decrease. To take this into account we would have to allow for the string to have a nonuniform density, and also a nonuniform tension. This would require us to introduce another functional freedom, describing the deviation of the string from its equilibrium density at each $x$ and $t$. The resulting system would be more complicated to handle.

This would be zero were it not for the slight difference in location where the slopes are evaluated. Since Newton's law equates this to $ma_y$ with $m$ of order $dx$, we only need to evaluate $F_y$ to this order. To do so we expand the first term in $dx$ keeping only the first order term. That is, we apply the relation $f(x+dx) = f(x) + f'(x)dx + O((dx)^2)$ to the first term. That is,

$$\frac{\partial y}{\partial x}\Big|_{x+dx} = \frac{\partial y}{\partial x}\Big|_{x} + \frac{\partial^2 y}{\partial x^2}\Big|_{x} dx + O((dx)^2), \tag{8}$$

so to $O(dx)$ $F_y$ becomes

$$F_y = T\frac{\partial^2 y}{\partial x^2}\, dx. \tag{9}$$

Now we impose Newton's law $F_y = ma_y$, with (**??**), (**??**), and (**??**). Note that both $F_y$ and $m$ are of order $dx$, while $a_y$ is of order 1, so the equation is sensible. Dividing by the common factor of $dx$ we obtain the string equation of motion

$$T\frac{\partial^2 y}{\partial x^2} = \mu\frac{\partial^2 y}{\partial t^2}. \tag{10}$$

Equivalently, we can write

$$\frac{\partial^2 y}{\partial t^2} - v^2\frac{\partial^2 y}{\partial x^2} = 0, \tag{11}$$

where $v = \sqrt{T/\mu}$. This is a wave equation for waves that travel at speed $v$.

**Exercise c**: Set $y(x,t) = f(x - vt)$, where $f$ is an arbitrary function of $x - vt$, and show that this function satisfies the wave equation (**??**) as long as $v^2 = T/\mu$. A sum of two such functions, one for each sign of $v$, yields the general solution. In this case dimensional analysis gave the precise wave speed, that is, the unknown dimensionless coefficient turned out to be 1.

Later in the course we will investigate more fully the nature and description of the solutions to the wave equation (**??**).