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Perturbation theory

From this book and most other books on mathematical physics you may have obtained the impression that most equations in the physical sciences can be solved. This is actually not true; most textbooks (including this book) give an unrepresentative state of affairs by only showing the problems that *can* be solved in closed form. It is an interesting paradox that as our theories of the physical world become more accurate, the resulting equations become more difficult to solve. In classical mechanics the problem of two particles that interact with a central force can be solved in closed form, but the three-body problem in which three particles interact has no analytical solution. In quantum mechanics, the one-body problem of a particle that moves in a potential can be solved for a limited number of situations only: for the free particle, the particle in a box, the harmonic oscillator, and the hydrogen atom. In this sense the one-body problem in quantum mechanics has no general solution. This shows that as a theory becomes more accurate, the resulting complexity of the equations makes it often more difficult to actually find solutions.

One way to proceed is to compute numerical solutions of the equations. Computers are a powerful tool and can be extremely useful in solving physical problems. Another approach is to find approximate solutions to the equations. In Chapter 12, scale analysis was used to drop from the equations terms that appear to be irrelevant. In this chapter, a systematic method is introduced to account for terms in the equations that are small but that make the equations difficult to solve. The idea is that a complex problem is compared to a simpler problem that can be solved in closed form, and to consider these small terms as a perturbation to the original equation. The theory of this chapter then makes it possible to determine how the solution is perturbed by the perturbation in the original equation; this technique is called *perturbation theory*. A classic reference on perturbation theory has been written by Nayfeh [74]. The book by Bender and Orszag [14] gives a useful and illustrative overview of a wide variety of perturbation methods.

The central idea of perturbation theory is introduced for an algebraic equation in Section 23.1. Sections 23.2, 23.3, and 23.5 contain important applications of perturbation theory to differential equations. As shown in Section 23.4, perturbation theory has a limited domain of applicability, and this may depend on the way the perturbation problem is formulated. Finally, it is shown in Section 23.7 that not every perturbation problem is well behaved; this leads to singular perturbation theory. Chapter 24 is devoted to the asymptotic evaluation of integrals.

23.1 Regular perturbation theory

As an introduction to perturbation theory let us consider the following equation

$$x^3 - 4x^2 + 4x = 0.01. \tag{23.1}$$

Let us for the moment assume that we do not know how to find the roots of a third order polynomial, so we cannot solve this equation. The problem is the small term 0.01 on the right-hand side. If this term were equal to zero, the resulting equation can be solved; $x^3 - 4x^2 + 4x = 0$ is equivalent to $x(x^2 - 4x + 4) = x(x-2)^2 = 0$, which has the solutions $x = 0$ and $x = 2$. In Figure 23.1 the polynomial of (23.1) is shown by the thick solid line; it is indeed equal to zero for $x = 0$ and $x = 2$.

The problem that we face is that the right-hand side of (23.1) is *not* equal to zero. In perturbation theory one studies the perturbation of the solution under a perturbation of the original equation. In order to do this, we replace the original equation (23.1) by the more general equation

$$x^3 - 4x^2 + 4x = \varepsilon. \tag{23.2}$$

When $\varepsilon = 0.01$ this equation is identical to the original problem, while for $\varepsilon = 0$ it reduces to the unperturbed problem that we can solve in closed form. It may appear

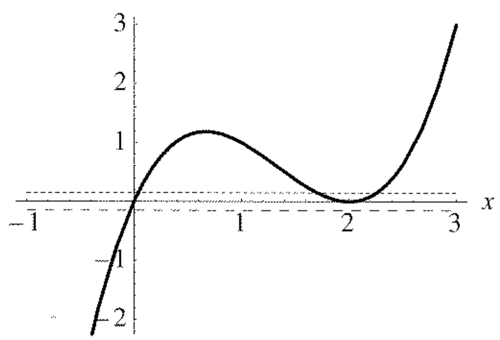

Fig. 23.1 The polynomial $x^3 - 4x^2 + 4x$ (thick solid line) and the lines $\varepsilon = 0.15$ (dotted line) and $\varepsilon = -0.15$ (dashed line).

that we have made the problem more complex because we still need to solve the same equation as our original equation, but it now contains a new variable ε as well! However, this is also the strength of this approach.

The solution of (23.2) is a function of ε so that

$$x = x(\varepsilon)\ . \tag{23.3}$$

In Section 3.1 the Taylor series was used to approximate a function $f(x)$ by a power series in the variable x:

$$f(x) = f(0) + x\frac{df}{dx}(x=0) + \frac{x^2}{2!}\frac{d^2 f}{dx^2}(x=0) + \cdots. \tag{3.11}$$

When the solution x of (23.2) depends in a regular way on ε, this solution can also be written as a similar power series by making the substitutions $x \to \varepsilon$ and $f \to x$ in (3.11):

$$x(\varepsilon) = x(0) + \varepsilon\frac{dx}{d\varepsilon}(\varepsilon=0) + \frac{\varepsilon^2}{2!}\frac{d^2 x}{d\varepsilon^2}(\varepsilon=0) + \cdots. \tag{23.4}$$

This expression is not very useful because we need the derivative $dx/d\varepsilon$ and higher derivatives $d^n x/d\varepsilon^n$ as well; in order to compute these derivatives we need to find the solution $x(\varepsilon)$ first, but this is just what we are trying to do. There is, however, another way to determine the series (23.4). Let us write the solution $x(\varepsilon)$ as a power series in ε

$$x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots. \tag{23.5}$$

The coefficients x_n are not known at this point, but once we know them the solution x can be found by inserting the numerical value $\varepsilon = 0.01$. In practice one truncates the series (23.5); it is this truncation that makes perturbation theory an approximation.

When the series (23.5) is inserted into (23.2) one needs to compute x^2 and x^3 when x is given by (23.5). Let us first consider the x^2-term. The square of a sum of terms is given by

$$\begin{aligned}(a+b+c+\cdots)^2 = a^2 + b^2 + c^2 + \cdots \\ + 2ab + 2ac + 2bc + \cdots.\end{aligned} \tag{23.6}$$

Let us apply this to the series (23.5) and retain only the terms up to order ε^2, this gives

$$\begin{aligned}\left(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots\right)^2 &= x_0^2 + \varepsilon^2 x_1^2 + \varepsilon^4 x_2^2 + \cdots \\ &\quad + 2\varepsilon x_0 x_2 + 2\varepsilon^2 x_0 x_2 + 2\varepsilon^3 x_1 x_2 + \cdots.\end{aligned} \tag{23.7}$$

If we are only are interested in retaining the terms up to order ε^2, the terms $\varepsilon^4 x_2^2$ and $2\varepsilon^3 x_1 x_2$ in this expression can be ignored. Collecting terms of equal powers of

ε then gives

$$\left(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots\right)^2 = x_0^2 + 2\varepsilon x_0 x_1 + \varepsilon^2\left(x_1^2 + 2x_0 x_2\right) + O(\varepsilon^3). \quad (23.8)$$

A similar expansion in powers of ε can be used for the term x^3. This expansion is based on the identity

$$\begin{aligned}(a + b + c + \cdots)^3 = {} & a^3 + b^3 + c^3 + \cdots \\ & + 3a^2 b + 3ab^2 + 3a^2 c + 3ac^2 + 3b^2 c + 3bc^2 + \cdots .\end{aligned} \quad (23.9)$$

Problem a Apply this identity to the series (23.5), collect together all the terms with equal powers of ε and show that up to order ε^2 the result is given by

$$\left(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots\right)^3 = x_0^3 + 3\varepsilon x_0^2 x_1 + 3\varepsilon^2\left(x_0 x_1^2 + x_0^2 x_2\right) + O(\varepsilon^3). \quad (23.10)$$

Problem b At this point we can express all the terms in (23.2) in a power series of ε. Insert (23.5), (23.8), and (23.10) into the original equation (23.2) and collect together terms of equal powers of ε to derive that

$$\begin{aligned}& x_0^3 - 4x_0^2 + 4x_0 \\ & + \varepsilon\left(3x_0^2 x_1 - 8x_0 x_1 + 4x_1 - 1\right) \\ & + \varepsilon^2\left(3x_0 x_1^2 + x_0^2 x_2 - 4x_1^2 - 8x_0 x_2 + 4x_2\right) + \cdots = 0 .\end{aligned} \quad (23.11)$$

In this and subsequent expressions the dots denote terms of order $O(\varepsilon^3)$. The term -1 in the term that multiplies ε comes from the right-hand side of (23.2).

At this point we use that ε does not have a fixed value, but that it can take any value within certain bounds. This means that expression (23.11) must be satisfied for a range of values of ε. This can only be the case when the coefficients that multiply the different powers ε^n are equal to zero. This means that (23.11) is equivalent to the following system of equations which consists of the terms that multiply the terms ε^0, ε^1 and ε^2 respectively:

$$\left.\begin{array}{ll} O(1)\text{-terms:} & x_0^3 - 4x_0^2 + 4x_0 = 0, \\ O(\varepsilon)\text{-terms:} & 3x_0^2 x_1 - 8x_0 x_1 + 4x_1 - 1 = 0, \\ O(\varepsilon^2)\text{-terms:} & 3x_0 x_1^2 + x_0^2 x_2 - 4x_1^2 - 8x_0 x_2 + 4x_2 = 0 . \end{array}\right\} \quad (23.12)$$

You may wonder whether we have not made the problem more complex. We started with a single equation for a single variable x, and now we have a system of coupled equations for many variables. However, we could not solve (23.2) for the single variable x, while it is not difficult to solve (23.12).

Problem c Show that (23.12) can be rewritten in the following form:

$$\left.\begin{array}{l} x_0^3 - 4x_0^2 + 4x_0 = 0, \\ \left(3x_0^2 - 8x_0 + 4\right) x_1 = 1, \\ \left(x_0^2 - 8x_0 + 4\right) x_2 = (4 - 3x_0) x_1^2 . \end{array}\right\} \tag{23.13}$$

The first equation is simply the unperturbed problem, this has the solutions $x_0 = 0$ and $x_0 = 2$. For reasons that will become clear in Section 23.7 we focus here on the solution $x_0 = 0$ only. Given x_0, the parameter x_1 follows from the second equation because this is a linear equation in x_1. The last equation is a linear equation in the unknown x_2 which can easily be solved once x_0 and x_1 are known.

Problem d Solve (23.13) in this way to show that the solution near $x = 0$ is given by

$$x_0 = 0, \qquad x_1 = \frac{1}{4}, \qquad x_2 = \frac{1}{16} . \tag{23.14}$$

Now we are close to the final solution of our problem. The coefficients of the previous expression can be inserted into the perturbation series (23.5) so that the solution as a function of ε is given by

$$x = 0 + \frac{1}{4}\varepsilon + \frac{1}{16}\varepsilon^2 + O(\varepsilon^3) . \tag{23.15}$$

At this point we can revert to the original equation (23.1) by inserting the numerical value $\varepsilon = 0.01$, which gives:

$$x = \frac{1}{4} \times 10^{-2} + \frac{1}{16} \times 10^{-4} + O(10^{-6}) = 0.002506 . \tag{23.16}$$

It should be noted that this is an approximate solution because the terms of order ε^3 and higher have been ignored. This is indicated by the term $O(10^{-6})$ in (23.16). Assuming that the error made by truncating the perturbation series is of the same order as the first term that is truncated, the error in the solution (23.16) is of the order 10^{-6}. For this reason the number on the right-hand side of (23.16) is given to six decimals; the last decimal is of the same order as the truncation error.

If this result is not sufficiently accurate for the application that one has in mind, then one can easily extend the analysis to higher powers ε^n in order to reduce the truncation error of the truncated perturbation series. Although the algebra resulting from doing this can be tedious, there is no reason why this analysis cannot be extended to higher orders.

A truly formal analysis of perturbation problems can be difficult. For example, the perturbation series (23.5) converges only for sufficiently small values of ε. It is often not clear whether the employed value of ε (in this case $\varepsilon = 0.01$) is sufficiently small to ensure convergence. Even when a perturbation series does not converge for a given value of ε, one can often obtain a useful approximation to the solution by truncating the perturbation series at a suitably chosen order [14]. In this case one speaks of an *asymptotic series*.

When one has obtained an approximate solution of a perturbation problem, one can sometimes substitute it back into the original equation to verify whether this solution indeed satisfies the equation with an acceptable accuracy. For example, inserting the numerical value $x = 0.002\,506$ in (23.1) gives

$$x^3 - 4x^2 + 4x = 0.009\,998\,9 = 0.01 - 0.000\,001\,1\,. \tag{23.17}$$

This means that the approximate solution satisfies (23.1) with a *relative* error that is given by $0.000\,001\,1/0.01 = 10^{-4}$. This is a very accurate result given the fact that only three terms were retained in the perturbation analysis of this section.

23.2 Born approximation

In many scattering problems one wants to account for the scattering of waves by the heterogeneities in the medium. Usually these problems are so complex that they cannot be solved in closed form. Suppose one has a background medium in which scatterers are embedded. When the background medium is sufficiently simple, one can solve the wave propagation problem for this background medium. For example, in Section 19.3 we computed the Green's function for the Helmholtz equation in a homogeneous medium.

In this section we consider the Helmholtz equation with a *variable* velocity $c(\mathbf{r})$ as an example of the application of perturbation theory to scattering problems. This means we consider the wave field $p(\mathbf{r}, \omega)$ in the frequency domain that satisfies the following equation:

$$\nabla^2 p(\mathbf{r}, \omega) + \frac{\omega^2}{c^2(\mathbf{r})} p(\mathbf{r}, \omega) = S(\mathbf{r}, \omega)\,. \tag{23.18}$$

In this expression $S(\mathbf{r}, \omega)$ denotes the source that generates the wave field. In order to facilitate a systematic perturbation analysis we decompose $1/c^2(\mathbf{r})$ into a term $1/c_0^2$ that accounts for a homogeneous reference model and a perturbation:

$$\frac{1}{c^2(\mathbf{r})} = \frac{1}{c_0^2}\left[1 + \varepsilon n(\mathbf{r})\right]. \tag{23.19}$$

In this expression ε is a small parameter which measures the strength of the heterogeneity. The function $n(\mathbf{r})$ gives the spatial distribution of the heterogeneity. Combining the previous expressions it follows that the wave field satisfies the show [3] that for elastic waves also the displacement is inversely proportional to $1/\sqrt{\rho c}$, where c is the propagation velocity of the elastic wave under consideration.)

The fact that the ground motion is inversely proportional to the square-root of the impedance is one of the factors that made the 1985 earthquake along the west coast of Mexico cause so much damage in Mexico City. This city is constructed on soft sediments which have filled the swamp onto which the city is built. The small value of the associated elastic impedance was one of the causes of the extensive damage in Mexico City after this earthquake.

23.7 Singular perturbation theory

In Section 23.1 we analyzed the behavior of the root of the equation $x^3 - 4x^2 + 4x = \varepsilon$ that was located near $x = 0$. As shown in that section, the unperturbed problem also has a root $x = 2$. The roots $x = 0$ and $x = 2$ can be seen graphically in Figure 23.1 because for these values of x the polynomial shown by the thick solid line is equal to zero. In Figure 23.1 the value $\varepsilon = +0.15$ is shown by a dotted line while the value $\varepsilon = -0.15$ is indicated by the dashed line. There is a profound difference between the two roots when the parameter ε is nonzero. The root near $x = 0$ depends in a continuous way on ε, and (23.2) has for the root near $x = 0$ a solution regardless of whether ε is positive or negative. This situation is completely different for the root near $x = 2$. When ε is positive (the dotted line), the polynomial has *two* intersections with the dotted line, whereas when ε is negative the polynomial does *not* intersect the dashed line at all. This means that depending on whether ε is positive or negative, the solution has two or zero solutions, respectively. This behavior cannot be described by a regular perturbation series of the form (23.5) because this expansion assigns *one* solution to each value of the perturbation parameter ε.

Let us first diagnose where the treatment of Section 23.1 breaks down when we apply it to the root near $x = 2$.

Problem a Insert the unperturbed solution $x_0 = 2$ into the second line of (23.13) and show that the resulting equation for x_1 is

$$0 \cdot x_1 = 1. \tag{23.83}$$

This equation obviously has no finite solution. This is related to the fact that the tangent of the polynomial at $x = 2$ is horizontal. First-order perturbation theory effectively replaces the polynomial by the straight line that is tangent to the polynomial. When this tangent line is horizontal, it can never have a value that is nonzero.

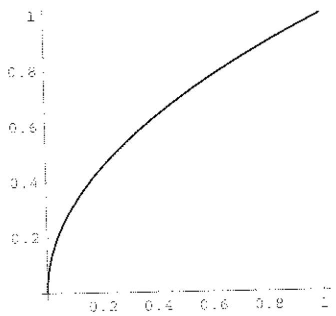

Fig. 23.4 Graph of the function $\sqrt{\varepsilon}$.

This means that the regular perturbation series (23.5) is not the appropriate way to study the behavior of the root near $x = 2$. In order to find out how this root behaves, let us set

$$x = 2 + y. \tag{23.84}$$

Problem b Show that under the substitution (23.84) the original problem (23.2) transforms to

$$y^3 + 2y^2 = \varepsilon. \tag{23.85}$$

We will not yet carry out a systematic perturbation analysis, but we will first determine the dependence of the solution y on the parameter ε. For small values of ε, the parameter y is also small. This means that the term y^3 can be ignored with respect to the term y^2. Under this assumption (23.85) is approximately equal to $2y^2 \approx \varepsilon$ so that $y \approx \sqrt{\varepsilon/2}$. This means that the solution does not depend on integer powers of ε as in the perturbation series (23.5), but that it does depend on the square-root of ε. The square-root of ε is shown in Figure 23.4. Note that for $\varepsilon = 0$ the tangent of this curve is vertical and that for $\varepsilon < 0$ the function $\sqrt{\varepsilon}$ is not defined for real values of ε.[†] This reflects the fact that the roots near $x = 2$ depend in a very different way on ε than the root near $x = 0$.

We know now that a regular perturbation series (23.5) is not the correct tool to use to analyze the root near $x = 2$. However, we do not yet know what type of perturbation series we should use for the root near $x = 2$; we only know that the perturbation depends to leading order on $\sqrt{\varepsilon}$. That is, let us make the following

[†] When one allows a complex solution $x(\varepsilon)$ of the equation, there are always two roots near $x = 2$. However, these complex solutions also display a fundamental change in their behavior when $\varepsilon = 0$, which is characterized by a bifurcation.

substitution:

$$x = 2 + \sqrt{\varepsilon}\, z. \tag{23.86}$$

Problem c Insert this solution into (23.2) and show that z satisfies the following equation:

$$\sqrt{\varepsilon} z^3 + 2z^2 = 1. \tag{23.87}$$

Now we have a new perturbation problem with a small parameter. However, this small parameter is not the original perturbation parameter ε, but it is the square-root $\sqrt{\varepsilon}$. The perturbation problem in this section is a *singular perturbation problem*. In a singular perturbation problem the solution is not a well-behaved function of the perturbation parameter. This has the result that the corresponding perturbation series cannot be expressed in powers ε^n, where n is a positive real integer. Instead, negative or fractional powers of ε are present in the perturbation series of a singular perturbation problem.

Problem d Since the small parameter in (23.87) is $\sqrt{\varepsilon}$, it makes sense to seek an expansion of z in this parameter:

$$z = z_0 + \varepsilon^{1/2} z_1 + \varepsilon z_2 + \cdots. \tag{23.88}$$

Collect together the coefficients of equal powers of ε when this series is inserted into (23.87) and show that this leads to the following equations for the coefficients z_0 and z_1:

$$\left.\begin{array}{ll} O(1)\text{-terms:} & 2z_0^2 - 1 = 0, \\ O(\varepsilon^{1/2})\text{-terms:} & z_0^3 + 4z_0 z_1 = 0. \end{array}\right\} \tag{23.89}$$

Problem e The first equation of (23.89) obviously has the solution $z_0 = \pm 1/\sqrt{2}$. Show that for both the plus and the minus signs $z_1 = -1/2$. Use these results to derive that the roots near $x = 2$ are given by

$$x = 2 \pm \frac{1}{\sqrt{2}}\sqrt{\varepsilon} - \frac{1}{2}\varepsilon + O(\varepsilon^{3/2}). \tag{23.90}$$

It is illustrative to compute the numerical values of these roots for the original problem (23.1), where $\varepsilon = 0.01$; this gives for the two roots:

$$x = 1.924 \quad \text{and} \quad x = 2.065. \tag{23.91}$$

In these numbers only three decimals are shown. The reason is that the error in the truncated perturbation series is of the order of the first truncated term, hence the error is of the order $(0.01)^{3/2} = 0.001$. When these solutions are compared with the perturbation solution (23.16) for the root near $x = 0$, it is striking that the singular perturbation series for the root near $x = 2$ converges much less rapidly than the regular perturbation series (23.16) for the root near $x = 0$. This is a consequence of the fact that the solution near $x = 2$ is a perturbation series in $\sqrt{\varepsilon}(= 0.1)$ rather than $\varepsilon(= 0.01)$. When the roots (23.91) are inserted into the polynomial (23.1) the following solutions are obtained for the two roots:

$$\left.\begin{array}{ll} x = 1.924: & x^3 - 4x^2 + 4x = 0.0111 = 0.01 + 0.0011. \\ x = 2.065: & x^3 - 4x^2 + 4x = 0.0087 = 0.01 - 0.0012. \end{array}\right\} \quad (23.92)$$

Note that these results are much less accurate than the corresponding result (23.1...) for the root near $x = 0$. Again this is a consequence of the singular behavior of the roots near $x = 2$.

The singular behavior of the roots of the polynomial (23.1) near $x = 2$ corresponds to the fact that the solution changes in a discontinuous way when the perturbation parameter ε goes to zero. It follows from Figure 23.1 that for the perturbation problem in this section the problem has one root near $x = 2$ when $\varepsilon = 0$, it has no roots when $\varepsilon < 0$ and there are two roots when $\varepsilon > 0$. Such a discontinuous change in the character of the solution also occurs in fluid mechanics in which the equation of motion is given by

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla \cdot (\rho\mathbf{v}\mathbf{v}) = \mu\nabla^2\mathbf{v} + \mathbf{F}. \quad (11.55)$$

In this expression the viscosity of the fluid gives a contribution $\mu\nabla^2\mathbf{v}$, where μ is the viscosity. This viscous term contains the highest spatial derivatives of the velocity that are present in the equation. When the viscosity μ goes to zero, the equation for fluid flow becomes a first order differential equation rather than a second order differential equation. This changes the number of boundary conditions that are needed for the solution, and hence it drastically affects the mathematical structure of the solution. This has the effect that boundary-layer problems are, in general singular perturbation problems [111].

When waves propagate through an inhomogeneous medium they may be focused onto focal points or focal surfaces [16]. These regions in space where the wave amplitude is large are called *caustics*. The formation of caustics depends on $\varepsilon^{2/3}$, where ε is a measure of the variations in the wave velocity [58, 102]. The non-integer power of ε indicates that the formation of caustics constitutes a singular perturbation problem.