

Falling sphere with air drag force

We considered the drag force and argued that to a good approximation it might depend just on the radius of the sphere R , the density of the air ρ , and the speed of the sphere through the air v . Only one combination of these three quantities has the dimensions of force:

$$F_{\text{drag}} \propto \rho R^2 v^2. \quad (1)$$

We can't infer the dimensionless coefficient. Let's define b as the product of this unknown coefficient times ρR^2 , so

$$F_{\text{drag}} = bv^2. \quad (2)$$

Note that the dimensions of b are M/L .

If the sphere has mass m and falls in a gravitational field with gravitational acceleration g , then Newton's law (taking the *down* direction as positive) says

$$m \frac{dv}{dt} = mg - bv^2. \quad (3)$$

If the sphere begins at rest it will initially accelerate with acceleration g . As it speeds up the drag force increases until it balances the gravity force. This will happen asymptotically, i.e. in an infinite amount of time. Let's work out the details.

To simplify the computation let's choose units so that $m = g = b = 1$. Can we really do this?! Yes, because there is no dimensionless combination of these three quantities. In particular, we can choose the units of mass, length, and time, as

$$m_0 = m \quad (4)$$

$$l_0 = m/b \quad (5)$$

$$t_0 = (m/bg)^{1/2} \quad (6)$$

Using these as our units, we have $m_0 = 1$, $l_0 = 1$, $t_0 = 1$, which implies that $m = b = g = 1$. The unit of length l_0 has an interesting interpretation: in a distance l_0 the falling sphere sweeps out a volume of air whose mass is $\sim \rho R^2 l_0 \sim b l_0$. According to the above definition of l_0 this is equal to m , the mass of the sphere. So this is a "natural" length scale for the problem.

It characterizes when the mass of the air is comparable to the mass of the sphere. It will also be the scale over which the sphere comes close to its terminal velocity.

With the above choice of units, the equation of motion for the sphere becomes

$$\frac{dv}{dt} = 1 - v^2 \quad (7)$$

or

$$\frac{dv}{1 - v^2} = dt. \quad (8)$$

Using the method of partial fractions this can be written

$$\frac{dv}{1 - v} + \frac{dv}{1 + v} = 2dt, \quad (9)$$

and integrating both sides then gives

$$\ln\left(\frac{1 + v}{1 - v}\right) = 2t + c_1. \quad (10)$$

If we choose $t = 0$ to be the time when the sphere is at rest $v = 0$, it follows that $c_1 = 0$, so exponentiating yields

$$\frac{1 + v}{1 - v} = e^{2t}. \quad (11)$$

Finally, solving for v we find

$$v = \frac{e^{2t} - 1}{e^{2t} + 1} = \frac{e^t}{e^t} \left(\frac{e^t - e^{-t}}{e^t + e^{-t}} \right) = \frac{\sinh t}{\cosh t}. \quad (12)$$

That is, we found the nice solution

$$v = \tanh t. \quad (13)$$

To find the position as a function of time we integrate the differential equation

$$\frac{dy}{dt} = \tanh t. \quad (14)$$

Rather than work this out “by hand”, let’s just look up the integral of $\tanh t$. This yields

$$y = \ln(\cosh t) + c_2. \quad (15)$$

If we choose the value of y at $t = 0$ to be 0, then $c_2 = 0$. This gives us the complete answer, expressed in terms of the units of mass m_0 , length l_0 and time t_0 .

How can we find the answer in arbitrary units, for example SI units? We don't have to re-do the whole problem. Rather, we just look at the answer, see where the factors of m_0 , l_0 and t_0 are hiding (invisibly since they are equal to 1 in these units!), and replace them by the expressions (6). So where are they hiding? Well, first note that $\cosh t$ contains different powers of t (e.g its expansion is $1 + t^2/2 + t^4/4! + \dots$). Since 1 is dimensionless all the terms must be so. Since t itself is time, which is not dimensionless, there must be a t_0 hiding there, i.e., we should replace t by t/t_0 . By this reasoning we infer that if had not set $m = g = b = 1$ initially, we would have found precisely t/t_0 here. Next, since $\cosh(t/t_0)$ is dimensionless and v has dimensions of velocity, we should multiply (13) by $l_0/t_0 = (mg/b)^{1/2}$. Similarly, in the result for the position y , we should multiply by l_0 . Hence the solution is given by

$$v = (l_0/t_0) \tanh(t/t_0) \tag{16}$$

and

$$y = l_0 \ln(\cosh(t/t_0)). \tag{17}$$

Now what about the interpretation of t_0 and l_0 ? Well from the form of the above solution for v , it's evident that t_0 is the time scale over which the velocity approaches terminal velocity. For instance, when $t = t_0$, the hyperbolic tangent is 0.76. When $t = 2t_0$, it's already 0.96, and when $t = 3t_0$ it's 0.995. Then from the solution for y we see that l_0 is roughly the distance traveled after a time t_0 has passed. More precisely, since $\ln(\cosh(1)) \simeq 0.4$, after a time t_0 the sphere has traveled a distance $\simeq 0.4l_0$.