Phase Transitions of Thermal Radiation in Anti-de Sitter Space

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In this presentation we review some results pertaining to black holes in Anti-de Sitter space following Hawking and Page (1983). We use units in which $G = c = \hbar = k_B = 1$. The metric of the covering space of anti-de Sitter space is of the form:

$$ds^2 = -V dt^2 + V^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \tag{1}$$

Here $V = 1 + \frac{r^2}{b^2}$ and $b = (-\frac{3}{\Lambda})^{1/2}$. Anti-de Sitter space is obtained by making the above space periodic in t, such that the point labeled by (t, r, θ, ϕ) is the same as that with the label $(t + \gamma, r, \theta, \phi)$; the value of γ is $2\pi b$.

We state without proof Hawking and Page's result that for zero mass particles of a conformally invariant field, the energy momentum tensor is

$$T_\nu^\mu = A\delta_\nu^\mu + f(T)V^{-2}(\delta_\nu^\mu - 4\delta_0^\mu \delta_\nu^0) \tag{2}$$

where $f(T) = \frac{\pi^2}{90} g T^4 + O(b^{-2} T^2)$ and g is the effective number of spin states. The first term represents an anomaly (the conformal anomaly), a symmetry that is broken in a quantum theory that is unbroken in the corresponding classical theory. Its volume integral over all space is formally infinite; we treat it as an unobservable renormalization of the vacuum energy.

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Einstein's equations have solutions which describe a Schwarzchild black hole in a spacetime which is asymptotically Anti-de Sitter. The metric in this case is the same as in (1), but with $V = 1 - \frac{2M}{r} + \frac{r^2}{b^2}$. We note that for both forms of V we have $R_{\mu\nu} = \Lambda g_{\mu\nu}$ and $R = 4\Lambda$.

The horizon occurs where $g_{00} = 0$, namely at $r = r_+$ where $V(r_+) = 0$. Explicitly,

$$r_+ = \frac{b^{2/3}}{3^{2/3}} \frac{(9M + \sqrt{81M^2 + 3b^2})^{2/3} - 3^{1/3} b^{2/3}}{(9M + \sqrt{81M^2 + 3b^2})^{1/3}} \tag{3}$$

and $r_+ \to 0$ as $M \to 0$. For this metric to be smooth and complete (i.e. prevent the occurrence of a conical singularity), we need t to be periodic with a period of $-i\beta$ where

$$\beta = \frac{4\pi b^2 r_+}{3r_+^2 + b^2} \tag{4}$$

or equivalently, we can let $\tau = it$ and say τ has period β; see the Appendix for the calculation. Now, the quantum amplitude for a state $|\varphi_1 >$ at time t_1 to propagate to a state $|\varphi_2 >$ at time t_2 is given by

$$< \varphi_2, t_2 |\varphi_1, t_1 >= \int D[\varphi] exp(iI[\varphi]) = \int D[\varphi] exp(-\hat{I}[\varphi]) \tag{5}$$

where we have taken a path integral over all matter fields φ that take the value φ_1 on the hypersurface $t = t_1$ and the value φ_2 at $t = t_2$, and $\hat{I} \equiv -iI$ is the Euclidean action. The quantity $\hat{I}$ is nonnegative for fields φ that are real on the Euclidean space (τ, r, θ, ϕ). Therefore the integral converges, a result that does not occur when evaluating the path integral on a Lorentzian space such as

(t, r, θ, ϕ). In the Schrodinger picture this amplitude is also given by

$$< \varphi_2 |exp(-iH(t_2 - t_1)| \varphi_1 > . \tag{6}$$

We can also take the path integral over all fields φ that are real on the Euclidean section and are periodic in the coordinate τ:

$$\int_{all\ real\ periodic\ \varphi} D[\varphi] exp(-\hat{I}[\varphi]) \equiv Z. \tag{7}$$

However, Z also corresponds to

$$Z = \sum_n < \varphi_n |exp(-iH(t_2 - t_1)| \varphi_n > \tag{8}$$

$$= \sum_n exp(-iE_n(t_2 - t_1)) < \varphi_n | \varphi_n > \tag{9}$$

where $|\varphi_n >$ are orthonormal eigenvectors of the Hamiltonian H. Therefore if we define $\beta \equiv i(t_2 - t_1)$ we get

$$Z = \sum_n exp(-\beta E_n), \tag{10}$$

which allows us to identify Z as the partition function of the canonical "ensemble" of fields φ. Or conversely, insisting that Z correspond to the partition function leads to $\beta = i\Delta t = \Delta \tau$. This construction associates a temperature $T = \beta^{-1}$ to the configuration of fields. In the case of the AdS Schwarzchild black hole, the period given in (4) is the inverse temperature of the black hole.

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However, we must be careful to note that the temperature actually measured locally by an observer, T_{loc}, is a frame-dependent quantity. The ratio of the temperatures measured by static observers at radii r_1 and r_2 is $T_2/T_1 = \chi(r_1)/\chi(r_2)$, χ being the norm of the time translation Killing vector field χ^a. This metric has the pleasant property that for both forms of V, $\chi^a = (1,0,0,0)$, up to a constant factor. Hence $|\chi^a\chi_a| = |-V| = V$ and $\chi(r) = V^{1/2}(r)$ so that

$$T_{loc} \propto \frac{1}{\chi(r)} = V^{-1/2}(r). \tag{11}$$

The constant of proportionality is β^{-1}, so

$$T_{loc}(r) = \beta^{-1}V^{-1/2}(r). \tag{12}$$

We see that $T_{loc} \to \infty$ as r approaches the horizon and $T_{loc} \to 0$ as r goes to infinity. The value of r at which $T_{loc} = \beta^{-1}$ is $(2b^2M)^{1/3}$. It is interesting to note that

$$\lim_{M\to\infty} r_+ = (2b^2M)^{1/3} \tag{13}$$

as well. But especially significant is that

$$\lim_{M\to\infty} \beta = \frac{4\pi}{3}2^{-1/3}b^{4/3}M^{-1/3}. \tag{14}$$

Thus, for large M, this black hole has $T \sim M^{1/3}$, quite different from the $T \sim M^{-1}$ behavior of black holes in asymptotically flat space (which is discussed

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briefly in the Appendix).

Next we consider the quantity $\hat{I}$ from (7). There are subtleties involved because the (Euclidean) action integral $\hat{I}$ is infinite when integrated over all space in the both the AdS case and the AdS-Schwarzchild case. However the *difference* in the two actions $(\hat{I}_{AdS-Schw.} - \hat{I}_{Schw})$ is actually the relevant quantity and turns out to be finite. The action is written

$$\hat{I} = -\frac{1}{16\pi} \int d^4x \sqrt{g}(R - 2\Lambda) \tag{15}$$

where R is the Ricci scalar. In general there is also a surface term in this action, but it vanishes in our case. The equation of motion (Einstein's equation) tells us $R = 4\Lambda$ (for both metrics!) which leads to

$$\hat{I} = -\frac{1}{8\pi}\Lambda \int d^4x \sqrt{g} \tag{16}$$

Thus we are left with the task of evaluating and subtracting the four-volumes of the two spaces. To achieve this we impose a "cutoff radius" r_c as the maximum value of the r coordinate, with the aim of taking the $r_c \to \infty$ limit as the final step. The four-volume of AdS spacetime is

$$v_1(r_c) = \int_0^{\beta'} d\tau \int_0^{r_c} r^2 dr \int d\Omega \tag{17}$$

and that for the black hole spacetime is

$$v_2(r_c) = \int_0^\beta d\tau \int_{r_+}^{r_c} r^2 dr \int d\Omega \qquad (18)$$

Witten (1998) explains that the black hole spacetime is smooth only if β is given by (4), but *any* value of β' is acceptable for smoothness of the AdS spacetime. The correct choice of β' gives β and β' the same proper coordinate value: $\sqrt{g_{00}^{(AdS)}}\beta' = \sqrt{g_{00}^{(bh)}}\beta$. Thus we need

$$\beta'\sqrt{1 + \frac{r_c^2}{b^2}} = \beta\sqrt{1 - \frac{2M}{r_c} + \frac{r_c^2}{b^2}}. \qquad (19)$$

This allows us to compute the difference in actions,

$$\hat{I}_0 \equiv \Delta\hat{I} = (\frac{-1}{8\pi})(\frac{-3}{b^2}) \lim_{r_c\to\infty} (v_2(r_c) - v_1(r_c)) = \frac{\pi r_+^2 (b^2 - r_+^2)}{b^2 + 3r_+^2}. \qquad (20)$$

We will use $\hat{I}_0$ to obtain Z later on.

By differentiating (4) with respect to r_+ we find that there is a maximum possible value of $\beta : 2\pi \ 3^{-1/2} \ b$, which corresponds to a minimum of T :

$$\frac{1}{2\pi}\sqrt{\frac{3}{b^2}} = \frac{1}{2\pi}\sqrt{-\Lambda} \equiv T_0. \qquad (21)$$

This value of T occurs when $r_+ = 3^{-1/2}b \equiv r_0$. It represents the minimum

temperature at which the black hole solution exists. Now, using (4)

$$r_+ = \frac{4b^2\pi \pm \sqrt{16\pi^2 b^4 - 12 b^2 \beta^2}}{6\beta}.$$

(22)

So there are two values of r_+ for each $T > T_0$. Using the definition of r_+ we can express M as

$$M = \frac{1}{2} r_+ (1 + \frac{r_+^2}{b^2})$$

(23)

and compute dM/dT using $dM/dT = (dM/dr_+)(dr_+/dT)$. As we shall see, it is very significant that $dM/dT > 0$ for $r_+ > r_0$ and $dM/dT < 0$ for $r_+ < r_0$.

Our discussion next leads to the gravitational properties of radiation itself, which can be treated as a perfect fluid with an equation of state $P = (1/3)\mu$ where μ is the energy density of the radiation. Anti-de Sitter space acts as a potential well which effectively confines most of the photons to a volume L^3 where L is on the order of b. Thus if more and more photons are added to the system, the mass becomes such that the corresponding r_+ given by (3) is greater than L, at which point the photon fluid collapses to form a black hole. This is completely analogous to the asymptotically flat case: a black hole appears when a mass distribution is confined to a radius less than the horizon distance, which is just $2M$ in the Schwarzchild case. The critical mass at which this occurs we call M_2; Hawking and page further claim that M_2 is of order b. This critical mass corresponds to a critical temperature T_2. But we know that the mass of the radiation within a ball of radius b goes like μb^3, and thermodynamics tells

us that $\mu \sim gT^4$ for radiation. Thus we have

$$\mu b^3 \sim M_2 \sim b, \; b^{-2} \sim \mu \sim gT_2^4 \tag{24}$$

and

$$T_2 \sim g^{-1/4}b^{-1/2}. \tag{25}$$

Returning to the partition function, we use Hawking and Page's result that $log \; Z = -\hat{I}_0$ so that

$$log \; Z = -\frac{\pi r_+^2 (b^2 - r_+^2)}{b^2 + 3r_+^2}. \tag{26}$$

This yields the expectation value of the energy:

$$< E >= -\frac{\partial}{\partial \beta} log \; Z = -\frac{d(log \; Z)}{dr_+}\frac{dr_+}{d\beta}. \tag{27}$$

Using (4), (23), and (26) the result is

$$< E >= \frac{1}{2}r_+(1 + \frac{r_+^2}{b^2}) = M. \tag{28}$$

Let $U \equiv < E >$ be considered the thermodynamic energy and $F = U - TS$ be the free energy of the black hole, where S is the black hole's entropy. From statistical mechanics, $F = -Tlog \; Z$ and $S = -\frac{\partial F}{\partial T}$. The entropy is thus given by

$$S = -\frac{d}{dT}(-Tlog \; Z) = log \; Z + \beta U = \pi r_+^2 = \frac{1}{4}A, \tag{29}$$

where A is the area of the event horizon; this is same form of entropy as in

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asymptotically flat space. From (23) we can deduce that for large M, $A \approx 4\pi(2b^2M)^{2/3}$ in contrast with $A \sim M^2$ in Schwarzchild spacetime. Since $S = log\,\mathcal{N}$, where $\mathcal{N}$ is the number of states, the density of states is proportional to $M^{-1/3}exp\,(\pi(2b^2M^{2/3}))$. The partition function,

$$Z = \int N(U)e^{-\beta U}dU = \int N(M)e^{-M/T}dM \tag{30}$$

goes like $\int M^{-1/3}exp\,(\alpha M^{2/3} - M/T)$ with constant α, which converges. This is indicative of a "well behaved" canonical ensemble. Such a situation does not occur in asymptotically flat space since there $N(M) \sim exp(4\pi M^2)$.

Now we return to equation (2), which pertains to radiation in AdS spacetime *without* a black hole. The constant term is unobservable, so we will take

$$T_\nu^\mu = f(T)V^{-2}(\delta_\nu^\mu - 4\delta_0^\mu \delta_\nu^0) \tag{31}$$

from here on. The energy U of the radiation is

$$U = \int T_{00}n^0 n^0 d^3x \tag{32}$$

where n^a is a unit vector orthogonal to the spatial hypersurface: $n^a = V^{-1/2}(1,0,0,0)$. Thus

$$U = \int g_{00}T_0^0 n^0 n^0 = \int 3V^{-2}f(T)d^3x. \tag{33}$$

Using $V = 1 + \frac{r^2}{b^2}$ yields

$$U = 3f(T) \int_0^\infty r^2 (1 + \frac{r^2}{b^2}) dr \int d\Omega = 3\pi^2 b^3 f(T) \approx \frac{\pi^4}{90} g b^3 T^4. \qquad (34)$$

This allows us to use

$$U = -\frac{\partial}{\partial \beta} log\ Z \rightarrow log\ Z = \int_0^T T^{-2} U dT \qquad (35)$$

to obtain $log\ Z \approx \frac{\pi^4}{90} g b^3 T^3$ for the radiation, and

$$F = -T log\ Z \approx -\frac{\pi^4}{90} g b^3 T^4. \qquad (36)$$

Hawking and Page are confident that the AdS spacetime and the AdS-Schwarzschild black hole are the only nonsingular positive-definite solutions of Einstein's equation that satisfy the periodic boundary conditions. Thus the radiation field will settle into one of two equilibrium states: thermal radiation on AdS space (no black hole) or AdS-Schwarzschild space, which represents a black hole. We have already determined that the latter option is not available (i.e. the solution does not exist) for $T < T_0$. Moreover, we concluded that the black hole is a certainty for $T > T_2$ (we are referring to the "high mass" black hole solution; see below). The following topic, then, will be the behavior of the system *between* these two limits.

Now we recall the observation that there are two values of r_+, and thus two black hole masses, associated with each temperature. As we saw, the higher

mass black hole has positive heat capacity, i.e. $dM/dT > 0$ while the lower one has $dM/dT < 0$. A black hole with negative heat capacity becomes hotter as its mass decreases. As it becomes hotter, its emission rate and thus the loss of its mass accelerates. Eventually it evaporates; this is what is believed to happen in asymptotically flat space. (The reverse process, in which the black hole becomes cooler and cooler as its mass increases indefinitely, is also conceivable.) However a black hole with positive heat capacity cools down as its mass decreases and heats up as its mass increases. Therefore, regardless of whether it begins in a hotter or cooler state than the ambient radiation, it will eventually come into equilibrium with the radiation. (However, most of the entropy will be contained in the black hole.) The upshot is that the high mass black hole is thermodynamically stable, like the "non-black hole"(thermal radiation on AdS space) whereas the low mass black hole is unstable.

We denote the free energy of the black hole solutions by F_{bh}:

$$F_{bh} = -T_{bh} \ log \ Z_{bh} = \frac{b^2 + 3r_+^2}{4\pi b^2 r_+} \frac{\pi r_+^2 (b^2 - r_+^2)}{b^2 + 3r_+^2} = \frac{r_+}{4}(1 - \frac{r_+^2}{b^2}), \qquad (37)$$

The low (high) mass black hole by F_l (F_h) and is obtained from F_{bh} by substituting the smaller (larger) value of r_+ for that temperature. This results in

$$F_l = \frac{T^3}{27}\left(2\pi b^2 - \sqrt{4\pi^2 b^4 - \frac{3b^2}{T^2}}\right)\left(\pi(-2\pi b^2 + \sqrt{4\pi^2 b^4 - \frac{3b^2}{T^2}}) + \frac{3}{T^2}\right), \ (38)$$

$$F_h = -\frac{T^3}{27}\left(2\pi b^2 + \sqrt{4\pi^2 b^4 - \frac{3b^2}{T^2}}\right)\left(\pi(2\pi b^2 + \sqrt{4\pi^2 b^4 - \frac{3b^2}{T^2}}) - \frac{3}{T^2}\right). \ (39)$$

The free energy of the non-black hole, will be written F_{AdS}; its form is given in (36). For consistency with the notation of Hawking and Page, we define T_1 to be the temperature above which F_h becomes negative, i.e. $F_h(T = T_1) = 0$. (F_l is never negative.) From (37), $T_1 = (\pi b)^{-1}$. The free energy of the high mass black hole state is negative for $T > T_1$ and becomes less than that of the non-black hole state when $T > \tilde{T}_1$. [Hawking and Page for some reason did not explicitly define $\tilde{T}_1$.] Obtaining an expression for $\tilde{T}_1$ would involve setting $F_h = F_{AdS}$; however this equation is transcendental in b. However we do know that $\tilde{T}_1$ is only slightly greater than T_1 if b is at least of order 1. We also note that $F_h \leq F_l$ with equality only at $T = T_0$. Furthermore, F_{bh} (low mass) $> F_{AdS}$ for all T. Thus for $T_0 < T < T_2$ we have two stable phases (non-black hole and high mass black hole) and one unstable phase (low mass black hole.) Any phase has a quantum amplitude to tunnel into the any other, but we expect the amplitude to tunnel into a state of higher free energy to be exponentially smaller than that of tunneling into a state of lower free energy. Thus we consider the following possibilities: the low mass black hole tunneling to either of the other two states for $T_0 < T < T_2$; the high mass black hole tunneling to the non-black hole state when $T_0 < T < \tilde{T}_1$; and non-black hole tunneling to the high mass black hole state when $\tilde{T}_1 < T < T_2$. The tunneling probability between two states takes the form

$$\Gamma \sim e^{-B} \tag{40}$$

where B is the difference in the actions of the two states at the same temperature.

We now turn to a purely (semi)classical analysis, restricting ourselves to phenomena expected without the consideration of quantum tunneling. First, the high mass black hole is thermodynamically stable, so once it forms, there is no *purely thermodynamic* impetus for it to, say, evaporate into the non-black hole state, even in the temperature range $T_0 < T < \tilde{T}_1$. The non-black hole state, consisting of radiation in equilibrium, is also thermodynamically stable. Therefore in the semiclassical picture it persists even when $\tilde{T}_1 < T < T_2$. But when $T > T2$ the state no longer exists and the radiation spontaneously collapses into a black hole on the high mass branch. Then there is the low mass black hole state. If it exists at a higher temperature than its surrounding radiation, semiclassically it will heat up, reducing its free energy in the process, and eventually evaporating. However, if it is cooler than its surrounding radiation, it gains mass and cools down even more. Intriguingly, this process actually *increases* its free energy F_l. This continues to happen until its temperature reaches T_0 at which point it enters the *high mass* black hole branch and stabilizes soon afterward. The addition of quantum considerations to this picture means that the tunneling transitions described above may alter this scheme over long time scales.

Thus we have reviewed a phase transition with three characteristic temperatures, T_0, $\tilde{T}_1$, and T_2 and discussed them in the context of the canonical ensemble. Hawking and Page also discuss this phenomenon with regard to the microcanonical ensemble; Witten elaborates (fifteen years later) on how this transition can be viewed from the perspective of a conformal field theory, per-

haps even in dimensions other than four. These topics unfortunately could not be addressed in a paper of this size, but indicate that this topic is rife with possibilities for further directions of exploration.

Appendix: Additional Calculations

The period of τ for the AdS-Schwarzchild metric is obtained by thinking of τ as an angular coordinate in $l - s$ space, where l is the proper radius $\int V^{-1/2} dr$ and s is the proper time $\int V^{1/2} d\tau$. The change in s when s has undergone one period is set equal to $2\pi l$ in the limit that $r \rightarrow r_+$, so that

$$\int_{\tau_1}^{\tau_2} V^{1/2}(r) d\tau = 2\pi \lim_{r \rightarrow r_+} \int_{r_+}^{r} V^{-1/2}(r') dr' \tag{41}$$

Now we make use of the fact that near r_+, $V(r) \approx V(r_+) + (r - r_+)V'(r_+) = (r - r_+)V'(r_+)$. Thus we have

$$(\tau_2 - \tau_1)V^{1/2} \equiv \beta V^{1/2} = 2\pi \lim_{r \rightarrow r_+} \int_{r_+}^{r} \frac{dr'}{\sqrt{(r' - r_+)V'(r_+)}} \tag{42}$$

Now $V'(r_+)$ is just $\frac{2M}{r_+^2} + \frac{2r_+}{b^2}$, giving

$$\beta \sqrt{(r - r_+)(\frac{2M}{r_+^2} + \frac{2r_+}{b^2})} = (2\pi)2\sqrt{(r - r_+)(\frac{2M}{r_+^2} + \frac{2r_+}{b^2})^{-1}} \tag{43}$$

$$\beta = 4\pi(\frac{2M}{r_+^2} + \frac{2r_+}{b^2})^{-1} \tag{44}$$

$$\rightarrow \beta = \frac{4\pi b^2 r_+}{3r_+^2 + b^2}. \tag{45}$$

Another option is to use $dl = V^{-1/2} dr$ and let $l = 0$ at the horizon $r = r_+$.

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The metric can now be written

$$ds^2 = V(l)d\tau^2 + dl^2.\qquad(46)$$

Next, $V(l) \approx V(0) + lV'(0) + \frac{1}{2}l^2V''(0)$, where primes now signify derivatives with respect to l. But $V'(0) = (dV/dr)(dr/dl)|_{l=0} = 0$ since $dr/dl = V$, which vanishes at the horizon. Since $V(0)$ vanishes as well, $V(l) \approx \frac{1}{2}l^2V''(0)$. So

$$ds^2 \approx l^2(\frac{1}{2}V''(0))d\tau^2 + dl^2\qquad(47)$$

The form of the metric is the same as that of polar coordinates in two dimensions, and thus the period of the variable $\sqrt{\frac{1}{2}V''(0)}\tau$ is just 2π ! This leads us to β, the period of τ:

$$\beta = \frac{2\pi}{\sqrt{\frac{1}{2}V''(0)}}.\qquad(48)$$

After some algebra, $\sqrt{\frac{1}{2}V''(0)} = \frac{1}{2}(\frac{2M}{r_+^2} + \frac{2r_+}{b^2})$ and we get

$$\beta = \frac{4\pi b^2 r_+}{3r_+^2 + b^2}.\qquad(49)$$

We also consider in more detail the form of the black hole temperature in asymptotically flat space. It is related to the surface gravity κ by $T = \frac{\kappa}{2\pi}$. From Wald (1984) we gather that

$$\kappa = \frac{\sqrt{M^2 - J^2/M^2 - Q^2}}{2M(M + \sqrt{M^2 - J^2/M^2 - Q^2}) - Q^2}.\qquad(50)$$

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For $M >> J, Q$ the result is $\kappa \to (4M)^{-1}$ and $T \to (8\pi M)^{-1}$.

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