Physics 776 Project

**Black Strings**

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It is a well known fact that four dimensional black holes are stable.That is, if we perform linear perturbations of the schwarzchild geometry,then such perturbations do not tend to grow in time.Rather, they are either radiated away to infinity ,or fall off into the black hole and decay.That the schwarzchild black hole is stable to linear perturbations,has been proven by the work of Vishweshwara[1],Moncrief[2]. Roughly, the reason to expect the stability of 4D black holes is because, the equations for perturbations of the schwarzchild black hole look like follows:

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial {r_*}^2} - V(r_*)f$$

This equation is written in terms of the wheeler coordinates $r_*$, and the perturbations being some scalar function f with compact support.The important point about the above equation is that the potential is positive semidefinite. This means the operator A ,defined by

$$A = -\frac{d^2}{d{r_*}^2} + V(r_*)$$

is a positive semidefinite self adjoint operator on the hilbert space $L_2(r_*)$.So in order to show stability of the 4D black hole, we need to show that ,for example given $C^\infty$ data for f with compact support at an instant of time, in $r_*$ ,then f remains bounded throughout time.Now ,we can see from the form of the equation above,that we can extract the following integral identity bounding the temporal derivative of f.

$\int(\dot{f}^2 + \bar{f}Af) = C$ where C is a constant. From the above equation it is easily seen that there cannot be any solution to the linearised equations which develop singularly.With a little more work(see Wald [3]) it can be shown that there are no solutions to the above equation that are unbounded.Hence,perturbations to the schwarzchild metric are bounded,and the 4D black hole is thus found to be stable classically. The black hole uniqueness theorems also indicate that 4D black holes are stable,because they tell us that the only effect of perturbations on the black hole geometry should be to either radiate away energy,or to change the parameters of the black hole (subject to the assumption of cosmic censorship).That this is in fact so, is concordant with the analysis of linear perturbations ,as discussed above.

We could ask the same question regarding higher dimensional analogues of black hole,the black string solutions.The answer turns out that black string solutions, unlike the black holes are unstable to linearised perturbations, as was first pointed out by Gregory and LaFlamme [4] .

## Black strings and the Gregory LaFlamme Instability

The simplest example of a black string is Sch $\times R$ with the metric

$$ds^2 = -(1 - \tfrac{2M}{r})dt^2 + (1 - \tfrac{2M}{r})^{-1}dr^2 \overset{1}{+} r^2 d\omega^2 + dz^2$$

The horizon topology is $S^2 \times R$ ,which justifies its name.In 5 dimensions,there is another black hole solution,with horizon topology $S^3$ .In fact, it turns out that in 5D the only possible topologies for the event horizon are $S^3$ and $S^2 \times S^1$ [5].

This is in contrast with the 4D case where the only possible topology of a stationary black hole horizon is the 2-sphere $S^2$ In the four dimensional case,this was shown first by Hawking[6] ,who observed that, if the topology of the event horizon was nonspherical,then it would be possible to deform the horizon outward slightly, and produce a 2-surface with negative convergence.This would imply that there was a trapped surface outside the event horizon, which would be a contradication with the assumed property of the spacetime viz. asymptotical predictivity,due to which the trapped surfaces had to be contained within the event horizon.The actual proof uses the Gauss Bonnet theorem to control the scalar curvature of the cross section of the horizon, and works only in 4d spacetime where the horizon cross section is 2D.However, the method of proof was later generalised in [6] where it was also shown that the 5D spacetime allowed the above mentioned topologies.

Before we investigate the question of stability of the black string ,let us go through an intuitive argument why it is expected that the black string might be unstable and eventually collapse to form a black hole.The argument involves comparing the entropies of a black string of length L and mass M ,to a black hole of the same mass M. A black hole of mass M has an entropy of $M^{3/2}$ while the black string has an entropy given by $M^2/L$.

This is because, at a given instant of time,the cross section of the event horizon of a stationary black hole is a 3-sphere,whose radius is ,say R.Then the "area" of the 3-sphere will be $R^3$ upto a numerical factor.Also, the area of the black string of length L and schwarzchild radius $r_+$ is $r_+^2 L$.However, the mass of the black string is (again ignoring numerical factors)given by $r_+ L$.Also, the mass of the 5D black hole will be ,upto numerical factors,given by $R^2$(The calculation of the ADM mass of a 5D black hole ,and a 5 D black string proceeds just the way the calculation of the 4D case goes,once we know the metric for the 5D black string and the metric for the 5D black hole.The metric for a 5D black hole is

$ds^2 = -(1-(\frac{R}{r})^2)dt^2 + (1-(\frac{R}{r})^2)^{-1}dr^2 + r^2 d\Omega^2$ where $d\Omega^2$ represents the volume of $S^3$ .With this caveat,the calculations follow) .So, putting it all together,we see that ,for a given mass, the black string has lesser entropy than a black hole ,when the schwarzchild radius of the black string $r_+$ is lesser than the length of the string L(upto numerical factors of order 1).

So,comparing the entropies,we find that the black hole has a greater entropy for a given mass than the black string ,when $L > M^{1/2}$ that is,there is a critical length of the order of the schwarzchild radius $R_s$ such that black strings of length greater than $L_c$ find it entropically favorable to coagulate into black holes.Of course,this argument is only heuristic,but it does indicate that there is a potential instability.In fact, such an instability was found by Gregory and LaFlamme.

In order to check perturbative stability, we need to perturb the metric ,and impose einstein's equations on the perturbation, with the background being the uniform black string metric.Then the perturbations will satisfy a second order PDE (the Lichnerowicz equation),whose solutions if they possess modes growing with time,for reasonable initial data would indicate instability.The subtleties in this procedure involve eliminating the gauge degrees of freedom, and imposing reasonable boundary conditions.Before we do this,let us go through a four dimensional argument which will tell us why,and when to expect an instability for the black string . To do this, observe that the perturbations to the black string geometry satisfy the Lichnerowicz equation:

$\nabla^2 h_{ab} + R_a{}^c{}_b{}^d h_{cd} = 0$

due to the symmetries of the problem, this equation can be written as :

$\nabla^2{}_4 h_{ab} - \partial^2{}_z h_{ab} = 0$

where the covariant derivative is with respect to the 4D schwarzchild metric.Expanding the perturbations in the z direction in fourier modes, we find that the equation takes the following form:

$\nabla^2{}_4 h_{ab} - \mu^2 h_{ab} = 0$

Now we see that this looks just like the 4D lichnerowicz equation for spherically symmetric 4D perturbations, except the tenor field is massive.Our first thought might be to conclude that ,since the 4D Lichnerowicz equation has no unstable modes, and so adding a mass term to such an equation cannot create any instability.This conclusion would be unjustified because, unlike the 4D case, here the tensor field $h_{ab}$ has three degrees of freedom, being massive, and so there is an extra propagating degree of freedom which can give rise to an instability, and in fact it is precisely such a mode (the s-wave perturbation corresponding to this degree of freedom) that gives rise to the Gregory-LaFlamme instability. Briefly, the analysis leading to these conclusions proceeds as follows.

substitute $g_{ab} \to g_{ab} + h_{ab}$ where $h_{ab}$ is a perturbation, and $g_a b$ is the background.In the DeDonder gauge the metric perturbations satisfy

$h_a{}^a = 0 = h^a{}_{a;b}$

the equation satisfied by the perturbations then becomes

$$\nabla^2 h_{ab} + R_a{}^c{}_b{}^d h_{cd} = 0$$

Here,the raising and lowering of indices is performed with the background metric.

The background metric is written in the form

$$ds^2 = -(1 - \frac{r_+}{r})dt^2 + (1 - \frac{r_+}{r})^{-1} dr^2 + r^2 d\omega^2 + dz^2$$

Even though the analysis below is specific to the 5D black string ,nothing changes in the analysis as you carry it through for the higher dimensional black strings.So the statement that black strings are unstable is quite generic,in a sense that'll be made precise later.

The transverse tracefree gauge does not fix all the gauge freedom.There is still an amount of gauge freedom corresponding to

$h_{ab} \to h_{ab} + \nabla_{(a}\xi_{b)}$ where $\xi_a$ is a vector field that is divergence free and harmonic(so that the gauge condition on the perturbations is preserved). The indices i are supposed to label the extra dimensions,which in our case labels the z direction. We also need to specify the boundary conditions on the PDE.For this purpose they placed cauchy data on an initial surface ending on the future event horizon, and initial perturbations had to be small at the horizon, with respect to the kruskal coordinates, and mode decomposition was perfomed in schwarzchild coordinates which were manifestly static. Because of the symmetries of the background spacetime ,we can divide the perturbations into scalar, vector and gravitational perturbations,

$$\begin{pmatrix} h_{\mu\nu} & h_{\mu i} \\ h_{j\nu} & h_{ij} \end{pmatrix}$$

It turns out that there are no scalar ,or vector perturbations that are unstable.So the only possibility is the tensor perturbations $h_{\mu\nu}$ .Also, here again we need to worry only about the s-wave perturbation of the form

$$h_{\mu\nu} = e^{\omega t} e^{\mu_i x^i} \begin{pmatrix} h_{tt} & h_{tr} & 0 & 0 \\ h_{tr} & h_{rr} & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & \frac{k}{sin^2\theta} \end{pmatrix}$$

Substituting this ansatz into the Lichnerowicz equation we see that the perturbations satisfy

$$\nabla^2 h_{\mu\nu} + \sum_i (\mu^i)^2 h_{\mu\nu} = 0$$

Using the extra gauge freedom available in the de-Donder gauge, we can set three of the four perturbations zero.Then we are left with a single equation in terms of ,say $h_{tr}$ which then obeys a second order ODE in terms of a single variable, r. It was for these modes that Gregory and LaFlamme found an instability.The instability arose for all modes with a wavelength greater than a critical wavelength, set by the horizon radius $r_+$

The nature of the instability was to make the horizon bulge at some points and contract at some other points in the fifth direction.This could roughly be viewed as the black string trying to redistribute its mass ,in order to gain entropy.

Although the analysis in [4] is only perturbative, the entropy arguments strongly suggest that the full nonlinear evolution of the 5D einstein's equations would lead to the fragmentation of the 5D black string into a periodic solution of little black holes which would subsequently merge to form a bigger 5D black hole which would be the entropically favored endpoint of this process. This viewpoint immediately implies that the process would violate cosmic censorship, because cosmic censorship does not allow bifurcation of the black hole horizon.The reason is as follows:

In a strongly asymptotically predictable spacetime, the null generators of the event horizon of a black hole (defined as the boundary of the past of future null infinity) can enter the horizon from outside ,but they cannot leave. So, if the horizon bifurcated, then we would have a situation where a connected compact set (the black hole region at time $t_1$) evolved to a region at time $t_2$ that was disconnected.That this is impossible follows from the fact that, if we trace the past of these disconnected sets $I^-(O_1)$ and $I^-(O_2)$, then they would necessarily have to meet $B(t_1)$ on disjoint sets themselves, contradicting the fact that at the earlier time, the black hole was a single connected component.The argument hinges on the fact that no point on $B(t_1)$ can be common to generator from $I^-(O_1)$ and $I^-(O_2)$ as that would violate the fact that the interior of the light cone is connected.So, if the black hole did in fact bifurcate at a later time, then we'd violate strong asymptotic predictability, and as a result cosmic censorship(for further technical details on this argument ,refer [10]).

The above argument can be evaded by asserting that the evolution of the black string will proceed towards smaller and smaller horizon areas until the effects of strong curvature become important,beyond which point the classical argument above does not hold,and the black string may then very well bifurcate into black holes as suggested.

That this probably does not occur , is the result of the arguments of Horowitz and Maeda[7].

**Fate of the Black String Instability**

The horizon cross section with a spacelike hypersurface determines an area which is topologically $S^2 \times R$ .The belief before Horowitz and Maeda's results[7] ,was that the presence of higher dimensional curvature could cause the $S^2$ to shrink in size ,as in the presence of negative energy densities, the area theorem in 4D would not hold.So there was no obstruction to the shrinking of the $S^2$. However, from the results of [7] it is clear that this viewpoint is too simplistic. Even though the 4D area theorem does not hold, there is still an obstruction to shrinking the $S^2$ , which comes from the fact that the 5D area theorem holds. Under reasonable assumptions (primarily that there are no naked singularities outside the horizon i.e. visible from $\mathcal{I}^+$ ) it can be shown that the 5D area theorem holds.That is ,the horizon area of a 5D black string cannot decrease with time.More precisely, if $\Sigma_{t_1}$ and $\Sigma_{t_2}$ are two partial cauchy surfaces for the spacetime,then the black hole region $J^+(B(t_1)) \subset B(t_2)$

Now this in itself severely constrains how the $S^2$ on the horizon can behave, as was noted by [7].If the horizon tried to shrink in the $S^2$ direction , then it would have to expand suitably in the transverse direction in order to keep the 5D area increasing, but this would lead to a shear which would itself try to shrink the horizon area further.In particular they showed that the $S^2$ on the horizon cannot shrink to zero size in finite affine parameter, and hence concluded that the black string cannot bifurcate in finite affine parameter (as a by product this implied that the null generators of the horizon remained complete and no violation of cosmic censorship occured in the Gregory Laflaame Instability).Let us go through their arguments more carefully.

Consider the Cauchy evolution of smooth initial data on a surface $\Sigma$ ,starting with infalling matter that produces a trapped surface and an apparent horizon with topology $S^2 \times S^1$ .For simplicity ,consider spherically symmetric collapse so that the resulting black hole is the uniform black string solution. Since the initial data is asymptotically flat, the maximal cauchy evolution will include at least part of $\mathcal{I}^+$ , and then from the assumption of no naked singularities outside the horizon ,we can infer that all trapped surfaces must be invisible from outside, that is ,there must be a non empty black string region defined as $\mathcal{M} - J^-(\mathcal{I}^+)$ .we are interested in studying the horizon of this black string . This event horizon is a 4D null surface parametrised by null geodesics.It will not be assumed that these null geodesics are future complete, in fact it will follow from the analysis. Since the initial data is assumed to be spherically symmetric, so will the evolved spacetime. so the metric of the cross section of the event horizon at a given instant of affine parameter $\lambda$ ,is given by

$ds^2 = e^{2\chi}dz^2 + e^{2\psi}d\Omega^2$ (call this $h_{\mu\nu}$ where the indices run between z, $\theta$ and $\phi$)

if the 2 sphere on the horizon shrinks to zero size in finite affine parameter, then $\psi$ goes to $-\infty$ . In the neighborhood of the horizon ,one can set up gaussian null coordinates, in which the metric looks like

$ds^2 = -f d\lambda^2 + 2drd\lambda + 2\beta d\lambda dz + h_{\mu\nu}dx^\mu dx^\nu$

where $f = 0$ on the horizon, $\partial_r f = 0$ on the horizon also along with $\beta = 0$. $\lambda$ is the affine parameter along the null generators of the horizon.

in this coordinate system the divergence of the null geodesic congruence is

$\theta = \dot{\chi} + 2\dot{\psi}$

where $\theta$ is defined as $\theta = h^{\mu\nu}\nabla_\mu l_\nu$ where $l_\nu$ is the tangent vector along the null normal congruence. Now we appeal to the Raychaudhri equation in 5 dimensions which reads

$\dot{\theta} = -\frac{\theta^2}{3} - \sigma^2 - R_{\mu\nu}l^\mu l^\nu$

for the metric above ,we find $\sigma^2 = 2\frac{(\dot{\chi}-\dot{\psi})^2}{3}$

The spacetime is Ricci-flat ,and so we can drop the last term in the raychaudhri equation.also, $\theta \geq 0$ ,and according to our assumptions the horizon slice in the $S^2$ direction is decreasing in area, that is $\dot{\psi} < 0$ . So we have

1.$\dot{\chi} \geq 2\dot{\psi}$

2.$\sigma_{\mu\nu}\sigma^{\mu\nu} \geq 6\dot{\psi}^2$

now we can integrate the focusing equation to get the following inequality, $\theta(\lambda) < \theta(\lambda_0) - 6\int_0^\lambda \dot{\psi}^2$ this implies that $\theta(\lambda) < 12\psi\lambda + 6\lambda + const$

that is, if $\psi$ goes to $-\infty$ in finite affine parameter, then by the above derivation, $\theta$ also goes to $-\infty$ and this leads to a contradiction. In fact ,the contradiction arises simply if any point on the horizon has $\theta < 0$ .This is because, if there is a point on the horizon ,where $\theta < 0$ then we can deform the horizon outward in the neighborhood of that point and end up with a deformed cross section whose future pointing null congruence is converging. But this means there is a trapped surface outside the event horizon, which is inconsistent with the asymptotic predictability of the spacetime(cf. Wald[8] for details) .

In fact Horowitz and Maeda [7] go ahead to prove that no $S^1$ on the horizon can shrink to zero size in finite affine parameter,by extending this argument.This argument also shows that the null generators of the horizon do in fact stay complete to the future ,as they extend to infinite affine parameter. So, there cannot be any singularities on the horizon. It has not been shown that there cannot be any pinch off for the horizon in infinite affine parameter.Horowitz and Maeda argue why it is reasonable to expect the horizon will not pinch off in infinite affine parameter.Lets go through their argument.

At some late time during the evolution ,consider the geometry of the spacetime near the smallest cross section of the horizon.The horizon cannot stay small for a distance much larger than its width, because then there will be further Gregory-LaFlamme instabilities, so if the geometry is evolving towards a stable solution, then the horizon area can remain small for a distance only on the order of its width.But in this case, the geometry of the spacetime near the neck will resemble that of two spherically symmetric black holes connected by a small neck, and it is known that in this case the apparent horizon moves out to encompass both black holes.So in this case, the event horizon will not be allowed to pinch off.

So, clearly we have ruled out the horizon from pinching off in finite affine parameter, and there are still restrictions of pinch off in infinite affine parameter,as the previous argument indicates.We know that ,whatever the final state evolves towards, it cannot be uniform,and in fact in the case where the black string is Sch $\times S^1$ we can say more: there cannot be any z independent solution with the proper radius of the circle decreasing in as we come in from infinity ,in such a manner that the horizon is still stable. This can be ruled out by performing a KK decomposition in the circle direction.We end up with a 4D schwarzchild black hole coupled to a scalar field and we know that 4D uniqueness theorems then imply that the scalar field is a constant. So the other possibility that is left,is that the black string settles down to a non-translationally invariant solution.If this happened, it would be an example where the most symmetric solution was unstable,and the system chose a broken symmetry phase (translation invariance in the circle direction being the symmetry that was broken) for its stable ground state.

Clearly, these results give new impetus for the search for stable non uniform solutions to the 5D vacuum einstein equations,that resemble the 5D black strings asymptotically.The problem reduces to solving a coupled set of nonlinear PDEs in two variables r and z.So far,there are no analytic solutions, but some work has been done numerically,which seems pretty suggestive.

As noted before,the Gregory-LaFlamme instability shows up as infrared effect,wherein all modes above a critical wavelength can destabilise the black string solution. If we work with a black string of topology $S^2 \times S^1$ then these instabilities will arise only for black strings whose schwarzchild radius is sufficiently small compared to the size of the circle.The compactification serves as an infrared regulator.So there will be a critical length for the size of the circle, where the instability of the uniform black string solution shows up.Expanding around this critical length to first order in nonuniformity, Gubser argued[9] based on the perturbative solution to einstein's equations ,that the transition between the uniform and inhomogenous black strings was probably a first order transition, and not continuous.

There is also been recent activity motivated by the Horowitz-Maeda paper, to numerically examine the evolution of the black string perturbations [11] ,and while there are no concrete results yet, what the authors of [11] seem to find ,is that the intermediate stage of evolution of the black string seems to be to a stage where the black string grows to a sequence of black holes connected by long thin necks.Beyond this point,it is not yet known what happens to the black string, as the code used by [10] breaks down ,due to certain harmless coordinate singularities.

## conclusion

In this review we have looked into the 5D black strings ,and studied the classical instability they possess.Recent work by Horowitz and Maeda [7] suggested that the endpoint of the black strings was probably a non translationally invariant state.Certainly, unlike what was thought before, the null generators of the horizon do not encounter any singularity during the process of the Gregory-LaFlamme instability.In view of this ,there has been a renewed interest in finding new static solutions describing non translationally invariant black strings, which could serve as the endpoint of the Gregory-LaFlamme instability.

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