A Report on the Noether Charge Approach to the 1st Law of Black Hole Mechanics

As an assignment of the 776 GR class
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Abstract

Here we have an overview of the Noether charge derivation of the first law of black hole mechanics, as described in [1-5]. The existence of a symmetry for the solutions is highlighted as a necessary condition for having only boundary contributions. Finally the example of two applications [6-7] is sketched and some future direction of research outlined.

1 Noether current and conserved quantities.

One of the benefits of the action formalism is the clear relation between symmetries of action and conserved quantities. In general, consider an action $\mathcal{S}$ as being a functional of fields ϕ, with appropriate tensor indices, and its derivatives $\nabla\phi$, where we use covariant derivatives compatible with a metric g_{ab} in the case of a curved background:

$$\mathcal{S} = \mathcal{S}[\phi] = \int \mathbf{L} = \int \mathcal{L}(\phi)\sqrt{g}d^4x. \tag{1}$$

Under a variation δ, we have:

$$\delta\mathcal{S} = \int \mathbf{E}_\phi \delta\phi + \int d\mathbf{\Theta}. \tag{2}$$

Where $\mathbf{E}_\phi = 0$ are the equations of motion, $\mathbf{\Theta} = \mathbf{\Theta}(\phi, \delta\phi)$ is a three-form function of the fields and linear in its variations and $\epsilon_{abcd} = \sqrt{g}\varepsilon_{abcd}$.

Example 1 *Consider a Klein-Gordon action on a background:*

$$\mathcal{S} = \int \nabla_a\phi\nabla_b\phi g^{ab}\sqrt{g}d^4x. \tag{3}$$

Its variation yields:

$$\delta\mathcal{S} = 2\int \nabla_a(\delta\phi\nabla_b\phi g^{ab})\sqrt{g}d^4x - 2\int \delta\phi\nabla^2\phi\sqrt{g}d^4x. \tag{4}$$

Recalling the relation between differential form and integrals:

$$\int A\sqrt{g}d^4x = \int A\sqrt{g}\epsilon_{abcd}dx^a \wedge dx^b \wedge dx^c \wedge dx^d = \int A\epsilon \tag{5}$$

by inspection, we see that:

$$\mathbf{E} = \nabla^2\phi\epsilon, \quad (6)$$
$$\mathbf{\Theta} = \delta\phi\nabla_a\phi g^{ab}\epsilon_{abcd}dx^b \wedge dx^c \wedge dx^d. \quad (7)$$

In general $\mathcal{S}$ can depend on the metric and its derivatives (in the case of a general theory of gravity). Also it is usually required for $\mathcal{S}$ to be diffemorphism invariant, which implies using tensors as the fundamental fields in such a way that $\mathcal{S}$ is a scalar. This kind of theory would be expressed by means of a Lagrangian 4-form:

$$\mathbf{L} = \mathbf{L}(g_{ab}, R_{abcd}, \nabla R, ...; \phi, \nabla\phi, ...), \quad (8)$$

and the equations of motion are given by

$$\delta\mathbf{L} = \mathbf{E}_g^{ab}\delta g_{ab} + \mathbf{E}_\phi\delta\phi + d\mathbf{\Theta}. \quad (9)$$

The expression of this diffeomorphism covariance turns out clear as one computes the Noether current generated by diffeomorphisms. Consider a vector field ξ defined on the spacetime $\mathcal{M}$. The flow of ξ induces a one-parameter family of diffeomorphisms from $\mathcal{M}$ to $\mathcal{M}$. The derivative of its action on tensors is the Lie derivative $\pounds_\xi$. Therefore

$$\pounds_\xi\mathbf{L} = \mathbf{E}\pounds_\xi\phi + d\mathbf{\Theta}(\phi, \pounds_\xi\phi).$$

Now we can define a conserved 3-form (dual to the Noether current vector) as

$$\mathbf{J}(\phi, \pounds_\xi\phi) = \mathbf{\Theta}(\phi, \pounds_\xi\phi) - \xi \cdot \mathbf{L}, \quad (10)$$

where the dot means the contraction of a form with a vector.

Eventually ξ will play the role of a Killing vector, and $\mathbf{J}$ will be related to the flux of energy or angular momentum. $\mathbf{J}$ is conserved in the sense that its divergent is 0 on shell, as in general

$$d\mathbf{J} = d\mathbf{\Theta} - d(\xi\cdot\mathbf{L}) = d\mathbf{\Theta} - \pounds_\xi\mathbf{L} =$$
$$= d\mathbf{\Theta} - (\mathbf{E}\pounds_\xi\phi + d\mathbf{\Theta}) = \mathbf{E}\pounds_\xi\phi. \quad (11)$$

When the fields satisfies the equations of motion, $\mathbf{E} = 0$, we have:

$$d\mathbf{J} = 0 \rightarrow \int_{\mathcal{N}} d\mathbf{J} = \int_{\partial\mathcal{N}} \mathbf{J},$$

where $\mathcal{N}$ is a 4-dimensional piece of $\mathcal{M}$. So the variation of $\mathbf{J}$ is determined by the influx of its current on the boundary, the characteristic of a conserved quantity.

As shown in [1] the form $\mathbf{J}$ is exact when the equations of motion hold. It provides a proof for the case where $\mathbf{J}$ can depend non linearly on the fields, and an explicit construction is given. Therefore,

$$\mathbf{J} = d\mathbf{Q}, \quad (12)$$

being $\mathbf{Q}$ the Noether charge. From this equation and the definition of $\mathbf{J}$ we can compute $\mathbf{Q}$ explicitly, but it is much easier to use the algorithm provided in [1].

Example 2 *Using the previous Klein-Gordon formulas:*

$$\mathbf{J}_{abc} = \pounds_\xi\phi i_\epsilon(\nabla\phi) - i_\epsilon\xi\nabla^2\phi = \quad (13)$$
$$= i_\epsilon\left(\pounds_\xi\phi\nabla\phi - \xi\nabla_a\phi\nabla^a\phi\right). \quad (14)$$

And from the cited algorithm we can compute:

$$\mathbf{Q}_{ab} = \epsilon_{abcd}\nabla^c\phi\xi^d\phi. \quad (15)$$

2 The 1st law of Black Hole Mechanics

Consider the Noether current $\mathbf{J} = \mathbf{\Theta}(\phi, \pounds_\xi \phi) - \xi \cdot \mathbf{L}$, with ϕ a solution of the equations of motion and ξ a vector field on $\mathcal{M}$. Consider a variation δ over solutions of $\mathbf{E} = 0$ but provided that $\delta\xi = 0$. Then

$$\begin{aligned} \delta\mathbf{J} &= \delta\mathbf{\Theta}(\phi, \pounds_\xi \phi) - \xi \cdot \delta\mathbf{L} = \\ &= [\delta\mathbf{\Theta}(\phi, \pounds_\xi \phi) - \pounds_\xi \mathbf{\Theta}(\phi, \delta\phi)] + d\left(\xi \cdot \mathbf{\Theta}(\phi, \delta\phi)\right). \end{aligned} \tag{16}$$

The bracket on the right hand is known as the symplectic current 2-form, that in general is defined by $\omega(\phi, \delta_1\phi, \delta_2\phi) = \delta_2\mathbf{\Theta}(\phi, \delta_1\phi) - \delta_1\mathbf{\Theta}(\phi, \delta_2\phi)$, it is antisymmetric and linear on the variations. So

$$\delta\mathbf{J} = \omega(\phi, \delta\phi, \pounds_\xi \phi) + d\left(\xi \cdot \mathbf{\Theta}(\phi, \delta\phi)\right). \tag{17}$$

This is a very general expression that, in the space of solutions, represents a differential relation between two nearby points of $\mathbf{E} = 0$.

To go further we need to make some restrictions. The goal is to recast the previous equation as an exact form, allowing us to express it only with quantities on the boundary of our manifold. First we consider variations that satisfy the linearized equations of motions, $\delta\mathbf{E} = 0$, this will imply that

$$\delta\mathbf{J} = \delta d\mathbf{Q} = d(\delta\mathbf{Q}). \tag{18}$$

The second assumption is to consider the vector field ξ as a Killing field. What means that we are only going to consider solutions of $\mathbf{E} = 0$ that posses a symmetry, $\pounds_\xi \phi = 0$, for all fields. Observe that the variation $\delta\phi$ need not to obey the symmetry. Therefore

$$\omega(\phi, \delta\phi, \pounds_\xi \phi) = 0, \tag{19}$$

the symplectic form vanishes if defined by the symmetry flow. Now we are left with

$$d\left(\delta\mathbf{Q}(\phi, \pounds_\xi \phi) - \xi \cdot \mathbf{\Theta}(\phi, \delta\phi)\right) = 0.$$

Integrating over a three dimensional slice Σ only the boundary terms will contribute, therefore

$$\int_{\partial\Sigma} [\delta\mathbf{Q} - \xi \cdot \mathbf{\Theta}(\phi, \delta\phi)] = 0. \tag{20}$$

As stated in [2], we can define the symplectic form

$$\mathbf{\Omega} = \int_\Sigma (\delta\mathbf{Q} - \xi \cdot \mathbf{\Theta}),$$

which, by definition, gives Hamilton's equations[1]

$$\delta\mathcal{H} = \mathbf{\Omega} = \int_\Sigma (\delta\mathbf{Q} - \xi \cdot \mathbf{\Theta}). \tag{21}$$

This implies that $\mathcal{H}$ can only be defined if we assume the existence of a 3-form $\mathbf{B}$ such that $\xi \cdot \mathbf{\Theta}(\phi, \delta\phi) = \delta(\xi \cdot \mathbf{B}(\phi))$, what would allow us to write

$$\int_{\partial\Sigma} \delta\left[\mathbf{Q} - \xi \cdot \mathbf{B}(\phi)\right] = 0. \tag{22}$$

[1]This is analogous to the usual classical mechanics formalism where $dH = i_\xi \Omega$ and $\Omega = d\theta$, where Ω is the symplectic 2-form, θ the symplectic current and $\xi = (\dot{p}, \dot{q})$

Now, in the context of black holes, $\partial\Sigma$ is the union of a collection of event horizons $\mathcal{H}$, which are generated by ξ and posses bifurcation surfaces $\mathcal{B}$, and the asymptotic infinity, in an asymptotically flat spacetime. It is important to mention that if we want all the horizons to be generated by the same Killing vector then we must impose that all the black holes have equal horizon angular velocity which will be equal to the orbital velocity.

Assuming we are working with globally hyperbolic spacetime, lets foliate it with slices that goes from the bifurcations point $\mathcal{B}$ to the infinity. At $\mathcal{B}$ it is clear the ξ vanishes, giving us

$$\delta \int_{\mathcal{B}} \mathbf{Q} = \delta \int_{\infty} (\mathbf{Q} - \xi \cdot \mathbf{B}), \tag{23}$$

again, where ξ is a symmetry of the solution. As said above, the quantity at the right hand is in fact the Hamiltonian of the theory, an integral evaluated at the infinity. It represents a variation of a conserved quantity like energy or angular momentum, depending of the definition of ξ.

In the case of a Kerr black hole, $\xi = \partial_t + \Omega_H \partial_\phi$, the horizon generator, ∂_t and ∂_ϕ being the coordinate basis vectors. So

$$\delta \int_{\infty} (\mathbf{Q} - \xi \cdot \mathbf{B}) = \delta\mathcal{E} - \Omega_H \mathcal{J}, \tag{24}$$

the variation of spacetime total energy and angular momentum. And the 1st law would read

$$\delta \frac{\kappa}{2\pi} S = \delta\mathcal{E} - \Omega_H \mathcal{J}, \tag{25}$$

where the identification $\int_{\mathcal{B}} \mathbf{Q} = \delta \frac{\kappa}{2\pi} S$ was made.

3 Solutions without symmetry

As it was stated previously, the existence of a symmetry for all dynamical variables, implies in a variation law dependent only on surfaces integrals. But it seems very restrictive to assume the existence of a full symmetry of the theory. On the other hand it is difficult to find solutions (of any theory) that have no continuous symmetry, what might justify the previous assumptions. But even if the physical fields of a theory respect a particular symmetry, that does not imply that all the fields on the action will. For example, this is the case for theories described by potentials, as the prefect fluid may be.

Example 3 *Suppose a vector physical quantity defined as the gradient of a scalar function. Then*

$$\mathbf{W} = dv, \quad 0 = \pounds_\xi \mathbf{w} = \pounds_\xi \mathbf{dv} = d < \xi, dv >, \tag{26}$$

what just implies that v is a linear function along the flow lines of ξ.

One way to approach this issue is to split the action into a part the obeys the symmetry and a part that does not. Details for a particular example is given in [6] but a general formulation is outlined in [4]. To sketch the procedure, first assume that $\mathbf{Q}$ can be defined off-shell by means of

$$\mathbf{J} + \mathbf{P}(\mathbf{E}, \xi, \phi) = d\mathbf{Q}, \tag{27}$$

where $\mathbf{P}$ is a 3-form that vanishes when the fields obey the equations of motion. The second step is to split the action. This is usually an ambiguous procedure that will influence the final interpretation of the quantities. Nevertheless we write

$$\mathbf{L}(\phi) = \mathbf{L}_g(g, R, \nabla R, ...) + \mathbf{L}_m(\psi, \nabla\psi, ..., g, R, \nabla R, ...), \tag{28}$$

and then follow the same steps shown on the previous section, computing respectively $\mathbf{\Theta}_g$ and $\mathbf{\Theta}_m$, $\mathbf{J}_g$ and $\mathbf{J}_m$. The key point is the symplectic 3-form $\omega(\phi, \pounds_\xi\phi, \delta\phi)$. For the fields that obey a symmetry ξ it vanishes as before, but for the fields that obey no symmetry, ω will give a non zero contribution of the form

$$\omega(\phi, \pounds_\xi\phi, \delta\phi) = d(\delta\mathbf{Q}_m(\phi, \pounds_\xi\phi) - \xi\cdot\mathbf{\Theta}_m(\phi,\delta\phi)) - \delta(\epsilon\cdot T\cdot\xi) + \frac{1}{2}\xi\cdot\epsilon T^{ab}\delta g_{ab}, \tag{29}$$

where the 3-form $(\epsilon\cdot T\cdot\xi)_{abc} = \epsilon_{abcd}T^{de}\xi_e$ and T^{ab} is the energy-momentum tensor, derived as the variation of the metric on the matter Lagrangian as usual. The Noether current and charges are related by

$$\begin{aligned}\mathbf{J} &= \delta(d\mathbf{Q}_m + d\mathbf{Q}_g),\\ \mathbf{J} &= \omega + d(\xi\cdot\mathbf{\Theta}_m + \xi\cdot\mathbf{\Theta}_g).\end{aligned}$$

Therefore we have

$$d\delta\mathbf{Q}_g - d\xi\cdot\mathbf{\Theta}_g = \frac{1}{2}\xi\cdot\epsilon T^{ab}\delta g_{ab} - \delta(\epsilon\cdot T\cdot\xi). \tag{30}$$

Considering the black hole setting, we may integrate over a 3-surface Σ that has bifurcation surfaces as the (disconnect) inner boundary and the asymptotic infinity as the outer boundary,

$$\begin{aligned}\int_{\mathcal{B}}\delta\mathbf{Q}_g &= \int_\infty \delta\left(\mathbf{Q}_g(\phi,\pounds_\xi\phi) - \xi\cdot\mathbf{B}_g\right) + \\ &\quad \int_\Sigma \left[\delta(\epsilon\cdot T\cdot\xi) + \frac{1}{2}\xi\cdot\epsilon T^{ab}\delta g_{ab}\right],\end{aligned} \tag{31}$$

where we assumed the existence of the 3-form $\mathbf{B}_g$ at infinity. The interpretation of the two surface integrals is not necessarily the one of entropy, energy and angular momentum, as in the case of theories with a symmetry vector field. On the other hand, if ξ is the asymptotic generator of time translations we can use

$$M_g = \int_\infty \left(\mathbf{Q}_g(\phi,\pounds_\xi\phi) - \xi\cdot\mathbf{B}_g\right) \tag{32}$$

as the definition of a gravitational mass. As clarified in [4], M_g only makes sense if the matter fields fall off fast enough so that the asymptotic solution of the full theory converges to the solution of the vacuum theory, in which we could define a vacuum gravitational mass that converges to M_g.

4 Final remarks

In the Noether charge approach it is clear that the first law of black hole mechanics connects stationary solutions of gravity plus matter, where the transition is done by means of variations that only need to obey the linearized equations of motion, no symmetry is required.

Some work has been done in finding stationary binary vacuum solutions, where the black holes have spin equal to the total angular momentum. Numerically, this problem was approached in [7]. There the authors use the existence of the helical Killing field to simplify the equations and come up with a family of stationary solutions for a binary black hole problem. Each of those solutions is parametrized by the distance between the horizons and a scaling parameter. As this is an approximated solution, the black holes just orbit in circular orbits. To make the connection of two solutions with different distance parameter, one need to impose some extra condition. They used the minimum entropy condition; the black hole areas would remain the same, therefore implying

$$\delta M = \Omega \delta J, \qquad \delta A_1 = \delta A_2 = 0. \tag{33}$$

From this family of solutions we can obtain a functional dependency between Ω and the space-time mass M. And they established the innermost circular orbit (ISCO) radius as the minimum point for $\Omega = \Omega(M)$.

In my point of view there are three natural directions to follow using the Noether charge approach. First to try to generalize the 1st law. The results in [4] and [6] only treat the perfect fluid, a system that has non-vanishing symplectic 2-form. Would be interesting to investigate new formulations for this system. Also if it is possible to by pass the necessity for $\delta\phi$ to be a solution of the linearized equations. A second investigation would be in direct application of the 1st law in approximate solutions to extract some information of the full solution like in [7]. Maybe with approximate radiative solutions would be possible to extract the signature of a merge, in the regime where the 1st law is valid. A third possible direction would be in the canonical quantum theory, trying to explore the off shell nature of **J** and the theory's physical phase space.

5 A few relations

There are many useful relations that helped deriving the above equations. Some of which are stated below:

$$\begin{aligned}
\delta\sqrt{g} &= \frac{1}{2}\sqrt{g}g^{ab}\delta g_{ab} = -\frac{1}{2}\sqrt{g}g_{ab}\delta g^{ab} \\
\pounds_\xi \omega &= d(\xi\cdot\omega) + \xi\cdot(d\omega) \\
d(V\cdot\epsilon) &= \nabla_a V^a \epsilon \\
\epsilon_{fabc}A^{[fe]}\omega_e &= \epsilon_{fe[bc}A^{fe}\omega_{a]}
\end{aligned} \tag{34}$$

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