Electromagnetism $(-+++)$:

$$S = \int -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \, d^4x + \int A_\mu j^\mu \, d^4x \, , \qquad \text{Gauge invariant : } \begin{array}{l} A_\mu \to A_\mu + \partial_\mu \theta \\ F_{\mu\nu} \to F_{\mu\nu} \end{array}$$

in flat space (Minkowskiy coords)

$$S = \int -\frac{1}{2} F^{0i} F_{0i} - \underbrace{\frac{1}{4} F^{ij} F_{ji}}_{\frac{1}{2} B^k B_k} \, , \qquad \partial_0 A_0 \text{ never appears} \longrightarrow \boxed{\pi^0 \equiv 0} \left(\frac{\partial L}{\partial \dot{A}_0} = 0 \right)$$

and $\left(\frac{\partial L}{\partial \dot{A}_i} \right)$: $\boxed{\pi^i(x) = \frac{\delta L}{\delta \dot{A}_i(x)} = -F^{0i}(x)}$ the electric field.

therefore :

$$H = \int \left\{ \cancel{\pi^0 \dot{A}_0}^{\,0} + \pi^i \dot{A}_i + \underbrace{\frac{1}{2} F^{0i} F_{0i}}_{-\pi^i \pi_i} + \frac{1}{2} B^2 \right\} d^4x =$$

$$\pi^i = \partial_0 A_i - \partial_i A_0 \, , \quad \dot{A}_i = \pi^i + \partial_i A_0$$

In general : $S[A] = \int F(A) \, dx$

$$\frac{\delta S}{\delta A(y)} = \int \left. \frac{\partial F}{\partial A} \right|_x \frac{\partial A(x)}{\partial A(y)} \, dx =$$

$$= \int \left. \frac{\partial F}{\partial A} \right|_x \delta(x-y) \, dx = \left. \frac{\partial F}{\partial A} \right|_y$$

$$= \int \left\{ \frac{1}{2} (\pi^2 + B^2) + \pi^i \partial_i A_0 \right\} d^4x = \int \underbrace{\frac{1}{2} (\pi^2 + B^2)}_{E^2 + B^2} - \underbrace{A_0 \partial_i \pi^i}_{A_0 \underbrace{(\partial_i \pi^i + \rho)}_{\text{gauss' law}}} + A_0 j^0 + A_i j^i$$

after imposing the constraints

gauss' law $\Rightarrow$ initial value constraint

See :

$$\partial_t (\nabla \cdot E - 4\pi \rho) = \nabla \cdot \partial_t E - 4\pi \partial_t \rho =$$

$\longleftarrow$ using Maxwell's eqn.

$$= \nabla \cdot (\nabla \times B) - \nabla \cdot (4\pi j) - 4\pi \partial_t \rho =$$

$$= -4\pi (\partial_t \rho + \nabla \cdot j) = 0 \quad \text{if charge is conserved.}$$

Canonical variables : $A_i(x) \, , \pi^i(x) = -E^i(x)$.