Recoil-Polarization Response Functions for Electroproduction of Pseudoscalar Mesons

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Recoil-polarization response functions for electroproduction of pseudoscalar mesons are derived in terms of CGLN and helicity amplitudes. Standard multipole expansions of the CGLN amplitudes are also used to express the response functions in terms of multipoles. These expansions are compared with those published by other authors.

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Introduction

In this notebook I derive the recoil-polarization response functions for electroproduction of pseudoscalar mesons from nucleons. Although I and my former student, Thomas Payerle, have produced copious notes about the formalism, the algebra is quite complex and there are many sections of Payerle's work which I had not verified previously. Furthermore, his work was limited to recoil polarization and I would like to extend our code, *epiprod*, to include target polarization also. Finally, there appear to be discrepancies between the multipole expansions published by the Mainz group, namely Drechsel and Tiator, and those published by the MIT group, Raskin and Donnelly, and used by Lourie in several papers and proposals. Therefore, in this notebook I develop an independent derivation of the response functions in terms of both CGLN and helicity amplitudes. I also develop tools to expand these functions in terms of multipoles subject to specified constraints on maximum angular momentum and/or dominance of particular multipoles. For the most part I will try to keep the conventions and notation as close to Payerle's as possible so that his work and the expressions coded in *epiprod* can be checked easily. Some effort is made to simplify important expressions, but it is often difficult to compel *Mathematica* to format expressions in the same way one does with more traditional derivations.

References

A.S. Raskin and T. W. Donnelly, *Polarization in Coincidence Electron Scattering from Nuclei*, Ann. Phys. (NY) **191**, 78 (1989).

R.W. Lourie *et al.*, *Investigation of the N → Δ Transition via Polarization Observables in Hall A*, TJNAF proposal 91-011.

R.W. Lourie, *Recoil Polarization Observables in Coincident Pion Electroproduction*, Nucl. Phys. **A509**, 653 (1990).

R. W. Lourie, *Quark Models and Polarized Electroproduction of the Roper Resonance*, Z. Phys. C **50**, 345 (1991).

D. Drechsel and L. Tiator, *Threshold Pion Photoproduction on Nucleons*, J. Phys. G **18**, 449 (1992).

T.M. Payerle, *User Manual for* ***epiprod*** *Version 2.3.6*, (UMd, 1993).

P. Dennery, *Theory of Electro- and Photoproduction of π Mesons*, Phys. Rev. **124**, 2000 (1961).

Initialization

Defaults and packages

```
ClearAll["Global`*"];
Off[General::spell, General::spell1];
$TextStyle = {FontFamily → "Times", FontSize → 12};

Needs["Utilities`Notation`"]

Symbolize[M₊]; Symbolize[M₋];
Symbolize[E₊]; Symbolize[E₋];
Symbolize[S₊]; Symbolize[S₋];
```

Manipulation of complex quantities

Throughout this notebook I assume that the only complex quantities are the electroproduction amplitudes themselves. Thus, it is useful to define rules and functions which simplify expressions in which the complex quantities are known to appear in specific patterns.

```
conjugate::usage =
    "A simple method for computing the
    conjugate of an object which is explicitly complex.";

conjugateRule = Complex[re_, im_] :> Complex[re, -im];
conjugate[exp__] := exp /. conjugateRule;
```

```
HermitianConjugate[A_] := Transpose[conjugate[A]]

trigToExp[ϕ_] = {Cos[ϕ] :> (ⅇ^(I ϕ) + ⅇ^(-I ϕ))/2, Sin[ϕ] :> (ⅇ^(I ϕ) - ⅇ^(-I ϕ))/(2 I)};

expToTrig[ϕ_] = {E^(Complex[0,a_] ϕ) → Cos[a ϕ] + I Sin[a ϕ]};

ContractAmplitudeProducts =
  {
  c_. A_a_ (A_a_)^* :> c Abs[A_a]^2,
  d_. + c_. A_a_ (B_b_)^* :> d + c (Re[A_a (B_b)^*] + I Im[A_a (B_b)^*]) /; b > a,
  d_. + c_. A_a_ (B_b_)^* :> d + c (Re[B_b (A_a)^*] - I Im[B_b (A_a)^*]) /; b < a,
  d_. Re[A_a_ (B_b_)^*] + e_. Re[C_c_ (B_b_)^*] :> d Re[(A_a + e/d C_c) (B_b)^*],
  d_. Im[A_a_ (B_b_)^*] + e_. Im[C_c_ (B_b_)^*] :> d Im[(A_a + e/d C_c) (B_b)^*]
  };

MyRules = {0^* → 0};
MyAssumptions = {α ∈ Reals, β ∈ Reals, γ ∈ Reals, δ ∈ Reals,
  θ ∈ Reals, ϕ ∈ Reals, ω > 0, q > ω, Q > 0, 0 < x < 1};
MySimplify = Simplify[# /. MyRules, MyAssumptions] &;
MyFullSimplify = FullSimplify[# /. MyRules, MyAssumptions] &;
```

■ Pauli matrices

```
σ⃗ = {{{0, 1}, {1, 0}}, {{0, -I}, {I, 0}}, {{1, 0}, {0, -1}}};

PauliAmp[A : {{_, _}, {_, _}}, 0] :=
 Module[{j}, 1/2 Sum[A[[j, j]], {j, 1, 2}]];
PauliAmp[A : {{_, _}, {_, _}}, i_] :=
 Module[{j}, 1/2 Sum[(σ⃗[[i]].A)[[j, j]], {j, 1, 2}]] /; i > 0
```

■ Basis vectors

The polar and azimuthal angles, θ and ϕ, describe the pion cm angle relative to the momentum transfer vector and the scattering plane.

```
p̂_π = {Sin[θ] Cos[ϕ], Sin[θ] Sin[ϕ], Cos[θ]};
q̂ = {0, 0, 1};
```

It is useful to formulate a very general basis for polarization vectors in terms of Euler angles.

```
rotz[θ_] := {{Cos[θ], Sin[θ], 0}, {-Sin[θ], Cos[θ], 0}, {0, 0, 1}};
rotx[θ_] := {{1, 0, 0}, {0, Cos[θ], Sin[θ]}, {0, -Sin[θ], Cos[θ]}};
roty[θ_] := {{Cos[θ], 0, -Sin[θ]}, {0, 1, 0}, {Sin[θ], 0, Cos[θ]}};
euler[α_, β_, γ_] := rotz[γ].roty[β].rotz[α];
```

The most useful basis for recoil polarization is normally the *ejectile basis* defined with $\hat{L}$ along the nucleon recoil momentum, $\hat{N} = \frac{\hat{q}\otimes\hat{L}}{|\hat{q}\otimes\hat{L}|}$ normal to the reaction plane, and $\hat{S} = \hat{N}\otimes\hat{L}$ within the reaction plane.

```
EjectileBasis = Thread[{Ŝ, N̂, L̂} → Transpose[euler[0, π + θ, π - ϕ]]]

{Ŝ → {Cos[θ] Cos[ϕ], Cos[θ] Sin[ϕ], -Sin[θ]},
 N̂ → {Sin[ϕ], -Cos[ϕ], 0}, L̂ → {-Cos[ϕ] Sin[θ], -Sin[θ] Sin[ϕ], -Cos[θ]}}
```

Similarly, the most useful basis for target polarization is the *target basis* defined with $\hat{L} = \hat{q}$ along the momentum transfer, $\hat{N} = \frac{\hat{q}\otimes\hat{p}}{|\hat{q}\otimes\hat{p}|}$ normal to the reaction plane, and $\hat{S} = \hat{N}\otimes\hat{L}$ within the reaction plane.

```
TargetBasis = Thread[{Ŝ, N̂, L̂} → Transpose[euler[0, 0, ϕ]]]

{Ŝ → {Cos[ϕ], -Sin[ϕ], 0}, N̂ → {Sin[ϕ], Cos[ϕ], 0}, L̂ → {0, 0, 1}}
```

Polarizations are determined here in the barycentric frame and can be transformed to the lab frame using a Wigner rotation later.

■ Helicity state vectors

The following notation for the phases is based upon Payerle's notes, although permitting 4 independent phases is clearly extravagant. Most results will be based upon the Jacob and Wick (JW) conventions for these phases.

$$\chi_{i,1} = e^{I\gamma}\{0, -1\};$$
$$\chi_{i,2} = e^{I(\gamma+\delta)}\{1, 0\};$$
$$\chi_i = \text{Transpose}[\{\chi_{i,1}, \chi_{i,2}\}];$$

$$\chi_{f,1} = e^{I\alpha}\left\{\text{Sin}\left[\frac{\theta}{2}\right] e^{-I\frac{\phi}{2}}, -\text{Cos}\left[\frac{\theta}{2}\right] e^{I\frac{\phi}{2}}\right\};$$
$$\chi_{f,2} = e^{I(\alpha+\beta)}\left\{\text{Cos}\left[\frac{\theta}{2}\right] e^{-I\frac{\phi}{2}}, \text{Sin}\left[\frac{\theta}{2}\right] e^{I\frac{\phi}{2}}\right\};$$
$$\chi_f = \text{Transpose}[\{\chi_{f,1}, \chi_{f,2}\}];$$

$$\text{phases}[JW] = \left\{\alpha \to \pi - \frac{\phi}{2}, \beta \to \pi + \phi, \gamma \to \pi, \delta \to \pi\right\};$$

■ Virtual photon polarization vectors

$$\vec{a} = \{a_x, a_y, a_z\};$$

```
rule[a_0] = {a_x → 0, a_y → 0, a_z → -Q/ω};
rule[a_1] = {a_x → -1/√2, a_y → -I/√2, a_z → 0};
rule[a_-1] = {a_x → 1/√2, a_y → -I/√2, a_z → 0};
```

Current operator in helicity representation

General form

The most general form of the current for pion electroproduction operator has been given by CGLN as follows.

$$\vec{J} = i\,\vec{\sigma}\,F_1 + F_2\,(\hat{p}_\pi.\vec{\sigma})\,\vec{\sigma}\otimes\hat{q} + \hat{p}_\pi\,(i\,\hat{q}\,.\vec{\sigma}\,F_3) + \hat{p}_\pi\,(i\,\hat{p}_\pi.\vec{\sigma}\,F_4) + \hat{q}\,(i\,\hat{q}\,.\vec{\sigma}\,F_5) + \hat{q}\,(i\,\hat{p}_\pi.\vec{\sigma}\,F_6)$$

In order to formulate this expression properly using *Mathematica*, it is useful to employ a generic vector $\vec{s}$ in place of $\vec{\sigma}$ first and to substitute after the vector operations have been evaluated. [Otherwise, the F_2 term in particular becomes troublesome.] The form below is expressed in the spin basis.

```
s⃗ = {s_x, s_y, s_z};

T[spin] = -I (I s⃗.a⃗ F_1 + F_2 (p̂_π.s⃗).(a⃗.Cross[s⃗, q̂]) + p̂_π.a⃗ (I q̂.s⃗ F_3) +
      p̂_π.a⃗ (I p̂_π.s⃗ F_4) + q̂.a⃗ (I q̂.s⃗ F_5) + q̂.a⃗ (I p̂_π.s⃗ F_6)) /.
    {s_x → σ⃗[[1]], s_y → σ⃗[[2]], s_z → σ⃗[[3]]} /. trigToExp[ϕ] // MySimplify
```

$$\Big\{\Big\{\frac{1}{2}\,e^{-i\phi}\,(\mathrm{Sin}[\theta]\,(i\,a_y\,(2\,F_2 - (-1 + e^{2\,i\,\phi})\,(F_3 + \mathrm{Cos}[\theta]\,F_4)) +$$
$$a_x\,(2\,F_2 + (1 + e^{2\,i\,\phi})\,(F_3 + \mathrm{Cos}[\theta]\,F_4))) +$$
$$2\,e^{i\,\phi}\,a_z\,(F_1 + \mathrm{Cos}[\theta]\,F_3 + \mathrm{Cos}[\theta]^2\,F_4 + F_5 + \mathrm{Cos}[\theta]\,F_6)),$$
$$\frac{1}{2}\,e^{-2\,i\,\phi}\,(-i\,a_y\,(2\,e^{2\,i\,\phi}\,F_1 - 2\,e^{2\,i\,\phi}\,\mathrm{Cos}[\theta]\,F_2 + (-1 + e^{2\,i\,\phi})\,\mathrm{Sin}[\theta]^2\,F_4) +$$
$$a_x\,(2\,e^{2\,i\,\phi}\,F_1 - 2\,e^{2\,i\,\phi}\,\mathrm{Cos}[\theta]\,F_2 + (1 + e^{2\,i\,\phi})\,\mathrm{Sin}[\theta]^2\,F_4) +$$
$$2\,e^{i\,\phi}\,\mathrm{Sin}[\theta]\,a_z\,(\mathrm{Cos}[\theta]\,F_4 + F_6))\Big\},$$
$$\Big\{\frac{1}{2}\,(i\,a_y\,(2\,F_1 - 2\,\mathrm{Cos}[\theta]\,F_2 - (-1 + e^{2\,i\,\phi})\,\mathrm{Sin}[\theta]^2\,F_4) +$$
$$a_x\,(2\,F_1 - 2\,\mathrm{Cos}[\theta]\,F_2 + (1 + e^{2\,i\,\phi})\,\mathrm{Sin}[\theta]^2\,F_4) +$$
$$2\,e^{i\,\phi}\,\mathrm{Sin}[\theta]\,a_z\,(\mathrm{Cos}[\theta]\,F_4 + F_6)),$$
$$\frac{1}{2}\,e^{-i\,\phi}\,(-\mathrm{Sin}[\theta]\,(-i\,a_y\,(2\,e^{2\,i\,\phi}\,F_2 + (-1 + e^{2\,i\,\phi})\,(F_3 + \mathrm{Cos}[\theta]\,F_4)) +$$
$$a_x\,(2\,e^{2\,i\,\phi}\,F_2 + (1 + e^{2\,i\,\phi})\,(F_3 + \mathrm{Cos}[\theta]\,F_4))) -$$
$$2\,e^{i\,\phi}\,a_z\,(F_1 + \mathrm{Cos}[\theta]\,F_3 + \mathrm{Cos}[\theta]^2\,F_4 + F_5 + \mathrm{Cos}[\theta]\,F_6))\Big\}\Big\}$$

The transition operator can now be transformed into the helicity basis.

```
T[helicity] = HermitianConjugate[χf].T[spin].χi // MySimplify
```

$$
\begin{aligned}
&\Big\{\Big\{\frac{1}{4}\, e^{-\frac{1}{2}\, i\, (2\,\alpha-2\,\gamma+3\,\phi)} \\
&\quad \Big(-2\,\mathrm{Sin}\Big[\frac{\theta}{2}\Big]\, a_x\, (2\, e^{2\, i\,\phi} F_1 + 2\, e^{2\, i\,\phi} F_2 + (1 + e^{2\, i\,\phi})\, (1 + \mathrm{Cos}[\theta])\, (F_3 + F_4)) + \\
&\qquad i\, \Big(2\,\mathrm{Sin}\Big[\frac{\theta}{2}\Big]\, a_y\, (2\, e^{2\, i\,\phi} F_1 + 2\, e^{2\, i\,\phi} F_2 + (-1 + e^{2\, i\,\phi})\, (1 + \mathrm{Cos}[\theta])\, (F_3 + F_4)) + \\
&\qquad\quad 4\, i\, e^{i\,\phi}\, \mathrm{Cos}\Big[\frac{\theta}{2}\Big]\, a_z\, (F_1 + \mathrm{Cos}[\theta]\, F_3 + \mathrm{Cos}[\theta]\, F_4 + F_5 + F_6)\Big)\Big), \\
&\quad \frac{1}{2}\, e^{i\,(\gamma+\delta) - \frac{1}{2}\, i\, (2\,\alpha+\phi)}\, \Big(-\mathrm{Cos}\Big[\frac{\theta}{2}\Big]\, (i\, a_y\, (2\, F_1 - 2\,\mathrm{Cos}[\theta]\, F_2 - (-1 + e^{2\, i\,\phi}) \\
&\qquad\qquad \mathrm{Sin}[\theta]^2\, F_4) + a_x\, (2\, F_1 - 2\,\mathrm{Cos}[\theta]\, F_2 + (1 + e^{2\, i\,\phi})\, \mathrm{Sin}[\theta]^2\, F_4) + \\
&\qquad\quad 2\, e^{i\,\phi}\, \mathrm{Sin}[\theta]\, a_z\, (\mathrm{Cos}[\theta]\, F_4 + F_6)) + \mathrm{Sin}\Big[\frac{\theta}{2}\Big] \\
&\qquad (\mathrm{Sin}[\theta]\, (i\, a_y\, (2\, F_2 - (-1 + e^{2\, i\,\phi})\, (F_3 + \mathrm{Cos}[\theta]\, F_4)) + \\
&\qquad\qquad a_x\, (2\, F_2 + (1 + e^{2\, i\,\phi})\, (F_3 + \mathrm{Cos}[\theta]\, F_4))) + \\
&\qquad\quad 2\, e^{i\,\phi}\, a_z\, (F_1 + \mathrm{Cos}[\theta]\, F_3 + \mathrm{Cos}[\theta]^2\, F_4 + F_5 + \mathrm{Cos}[\theta]\, F_6))\Big)\Big\}, \\
&\Big\{\frac{1}{4}\, e^{-\frac{1}{2}\, i\, (2\,\alpha+2\,\beta-2\,\gamma+3\,\phi)}\, \Big(-2\,\mathrm{Cos}\Big[\frac{\theta}{2}\Big]\, a_x\, (2\, e^{2\, i\,\phi} F_1 - 2\, e^{2\, i\,\phi} F_2 + \\
&\qquad (1 + e^{2\, i\,\phi})\, (-1 + \mathrm{Cos}[\theta])\, (F_3 - F_4)) + i\, \Big(2\,\mathrm{Cos}\Big[\frac{\theta}{2}\Big]\, a_y \\
&\qquad\ (2\, e^{2\, i\,\phi} F_1 - 2\, e^{2\, i\,\phi} F_2 + (-1 + e^{2\, i\,\phi})\, (-1 + \mathrm{Cos}[\theta])\, (F_3 - F_4)) - \\
&\qquad 4\, i\, e^{i\,\phi}\, \mathrm{Sin}\Big[\frac{\theta}{2}\Big]\, a_z\, (F_1 + \mathrm{Cos}[\theta]\, F_3 - \mathrm{Cos}[\theta]\, F_4 + F_5 - F_6)\Big)\Big), \\
&\quad \frac{1}{4}\, e^{-\frac{1}{2}\, i\, (2\,\alpha+2\,\beta-2\,\gamma-2\,\delta+\phi)}\, \Big(2\, i\,\mathrm{Sin}\Big[\frac{\theta}{2}\Big]\, a_y \\
&\qquad (2\, F_1 + 2\, F_2 - (-1 + e^{2\, i\,\phi})\, (1 + \mathrm{Cos}[\theta])\, (F_3 + F_4)) + \\
&\quad\ 2\,\mathrm{Sin}\Big[\frac{\theta}{2}\Big]\, a_x\, (2\, F_1 + 2\, F_2 + (1 + e^{2\, i\,\phi})\, (1 + \mathrm{Cos}[\theta])\, (F_3 + F_4)) + \\
&\quad\ 4\, e^{i\,\phi}\, \mathrm{Cos}\Big[\frac{\theta}{2}\Big]\, a_z\, (F_1 + \mathrm{Cos}[\theta]\, F_3 + \mathrm{Cos}[\theta]\, F_4 + F_5 + F_6)\Big)\Big\}\Big\}
\end{aligned}
$$

■ Linear combination of CGLN amplitudes

```
FtoℱF = {F1 → ℱ1, F2 → ℱ2, F3 → ℱ3,
   F4 → ℱ4, F5 → ℱ5 - ℱ1 - Cos[θ] ℱ3, F6 → ℱ6 - ℱ4 Cos[θ]};
```

```
ℱtoF = {ℱ1 → F1, ℱ2 → F2, ℱ3 → F3,
   ℱ4 → F4, ℱ5 → F1 + Cos[θ] F3 + F5, ℱ6 → F4 Cos[θ] + F6};
```

```
T[helicity] /. Fto𝓕 // MySimplify
```

$$\Big\{\Big\{\frac{1}{4}\,e^{-\frac{1}{2}\,i\,(2\,\alpha-2\,\gamma+3\,\phi)}\Big(-2\,\mathrm{Sin}\big[\tfrac{\theta}{2}\big]\,a_x\,(2\,e^{2\,i\,\phi}\,\mathcal{F}_1+2\,e^{2\,i\,\phi}\,\mathcal{F}_2+(1+e^{2\,i\,\phi})\,(1+\mathrm{Cos}[\theta])\,(\mathcal{F}_3+\mathcal{F}_4))+i\Big(2\,\mathrm{Sin}\big[\tfrac{\theta}{2}\big]\,a_y\,(2\,e^{2\,i\,\phi}\,\mathcal{F}_1+2\,e^{2\,i\,\phi}\,\mathcal{F}_2+(-1+e^{2\,i\,\phi})\,(1+\mathrm{Cos}[\theta])\,(\mathcal{F}_3+\mathcal{F}_4))+4\,i\,e^{i\,\phi}\,\mathrm{Cos}\big[\tfrac{\theta}{2}\big]\,a_z\,(\mathcal{F}_5+\mathcal{F}_6)\Big)\Big),\ \frac{1}{4}\,e^{-\frac{1}{2}\,i\,(2\,\alpha-2\,\gamma-2\,\delta+\phi)}\Big(-2\,\mathrm{Cos}\big[\tfrac{\theta}{2}\big]\,a_x\,(2\,\mathcal{F}_1-2\,\mathcal{F}_2+(1+e^{2\,i\,\phi})\,(-1+\mathrm{Cos}[\theta])\,(\mathcal{F}_3-\mathcal{F}_4))+i\Big(2\,\mathrm{Cos}\big[\tfrac{\theta}{2}\big]\,a_y\,(-2\,\mathcal{F}_1+2\,\mathcal{F}_2+(-1+e^{2\,i\,\phi})\,(-1+\mathrm{Cos}[\theta])\,(\mathcal{F}_3-\mathcal{F}_4))-4\,i\,e^{i\,\phi}\,\mathrm{Sin}\big[\tfrac{\theta}{2}\big]\,a_z\,(\mathcal{F}_5-\mathcal{F}_6)\Big)\Big)\Big\},\ \Big\{\frac{1}{4}\,e^{-\frac{1}{2}\,i\,(2\,\alpha+2\,\beta-2\,\gamma+3\,\phi)}\Big(-2\,\mathrm{Cos}\big[\tfrac{\theta}{2}\big]\,a_x\,(2\,e^{2\,i\,\phi}\,\mathcal{F}_1-2\,e^{2\,i\,\phi}\,\mathcal{F}_2+(1+e^{2\,i\,\phi})\,(-1+\mathrm{Cos}[\theta])\,(\mathcal{F}_3-\mathcal{F}_4))+i\Big(2\,\mathrm{Cos}\big[\tfrac{\theta}{2}\big]\,a_y\,(2\,e^{2\,i\,\phi}\,\mathcal{F}_1-2\,e^{2\,i\,\phi}\,\mathcal{F}_2+(-1+e^{2\,i\,\phi})\,(-1+\mathrm{Cos}[\theta])\,(\mathcal{F}_3-\mathcal{F}_4))-4\,i\,e^{i\,\phi}\,\mathrm{Sin}\big[\tfrac{\theta}{2}\big]\,a_z\,(\mathcal{F}_5-\mathcal{F}_6)\Big)\Big),\ \frac{1}{4}\,e^{-\frac{1}{2}\,i\,(2\,\alpha+2\,\beta-2\,\gamma-2\,\delta+\phi)}\Big(2\,i\,\mathrm{Sin}\big[\tfrac{\theta}{2}\big]\,a_y\,(2\,\mathcal{F}_1+2\,\mathcal{F}_2-(-1+e^{2\,i\,\phi})\,(1+\mathrm{Cos}[\theta])\,(\mathcal{F}_3+\mathcal{F}_4))+2\,\mathrm{Sin}\big[\tfrac{\theta}{2}\big]\,a_x\,(2\,\mathcal{F}_1+2\,\mathcal{F}_2+(1+e^{2\,i\,\phi})\,(1+\mathrm{Cos}[\theta])\,(\mathcal{F}_3+\mathcal{F}_4))+4\,e^{i\,\phi}\,\mathrm{Cos}\big[\tfrac{\theta}{2}\big]\,a_z\,(\mathcal{F}_5+\mathcal{F}_6)\Big)\Big\}\Big\}$$

■ Transition matrices with general phases

```
T0 = (T[helicity] /. rule[a0] /. Fto𝓕 /. trigToExp[ϕ]) // Simplify
```

$$\Big\{\Big\{\frac{e^{-\frac{1}{2}\,i\,(2\,\alpha-2\,\gamma+\phi)}\,Q\,\mathrm{Cos}[\frac{\theta}{2}]\,(\mathcal{F}_5+\mathcal{F}_6)}{\omega},\ \frac{e^{-\frac{1}{2}\,i\,(2\,\alpha-2\,\gamma-2\,\delta-\phi)}\,Q\,\mathrm{Sin}[\frac{\theta}{2}]\,(-\mathcal{F}_5+\mathcal{F}_6)}{\omega}\Big\},\ \Big\{-\frac{e^{-\frac{1}{2}\,i\,(2\,\alpha+2\,\beta-2\,\gamma+\phi)}\,Q\,\mathrm{Sin}[\frac{\theta}{2}]\,(\mathcal{F}_5-\mathcal{F}_6)}{\omega},\ -\frac{e^{-\frac{1}{2}\,i\,(2\,\alpha+2\,\beta-2\,\gamma-2\,\delta-\phi)}\,Q\,\mathrm{Cos}[\frac{\theta}{2}]\,(\mathcal{F}_5+\mathcal{F}_6)}{\omega}\Big\}\Big\}$$

```
T1 = (T[helicity] /. rule[a1] /. Fto𝓕 /. trigToExp[ϕ]) // Simplify
```

$$\Big\{\Big\{\frac{e^{-\frac{1}{2}\,i\,(2\,\alpha-2\,\gamma-\phi)}\,\mathrm{Sin}[\frac{\theta}{2}]\,(2\,\mathcal{F}_1+2\,\mathcal{F}_2+(1+\mathrm{Cos}[\theta])\,(\mathcal{F}_3+\mathcal{F}_4))}{\sqrt{2}},\ \frac{e^{-\frac{1}{2}\,i\,(2\,\alpha-2\,\gamma-2\,\delta-3\,\phi)}\,\mathrm{Sin}[\frac{\theta}{2}]\,\mathrm{Sin}[\theta]\,(-\mathcal{F}_3+\mathcal{F}_4)}{\sqrt{2}}\Big\},\ \Big\{\frac{e^{-\frac{1}{2}\,i\,(2\,\alpha+2\,\beta-2\,\gamma-\phi)}\,\mathrm{Cos}[\frac{\theta}{2}]\,(2\,\mathcal{F}_1-2\,\mathcal{F}_2+(-1+\mathrm{Cos}[\theta])\,(\mathcal{F}_3-\mathcal{F}_4))}{\sqrt{2}},\ -\frac{e^{-\frac{1}{2}\,i\,(2\,\alpha+2\,\beta-2\,\gamma-2\,\delta-3\,\phi)}\,(1+\mathrm{Cos}[\theta])\,\mathrm{Sin}[\frac{\theta}{2}]\,(\mathcal{F}_3+\mathcal{F}_4)}{\sqrt{2}}\Big\}\Big\}$$

```
T-1 = (T[helicity] /. rule[a-1] /. Fto𝓕 /. trigToExp[ϕ]) // Simplify
```

$$\left\{\left\{-\frac{e^{-\frac{1}{2} i (2\alpha-2\gamma+3\phi)} (1+\text{Cos}[\theta]) \text{Sin}[\frac{\theta}{2}] (\mathcal{F}_3+\mathcal{F}_4)}{\sqrt{2}}, -\frac{e^{-\frac{1}{2} i (2\alpha-2\gamma-2\delta+\phi)} \text{Cos}[\frac{\theta}{2}] (2\mathcal{F}_1-2\mathcal{F}_2+(-1+\text{Cos}[\theta])(\mathcal{F}_3-\mathcal{F}_4))}{\sqrt{2}}\right\}, \left\{-\frac{e^{-\frac{1}{2} i (2\alpha+2\beta-2\gamma+3\phi)} \text{Cos}[\frac{\theta}{2}] (-1+\text{Cos}[\theta]) (\mathcal{F}_3-\mathcal{F}_4)}{\sqrt{2}}, \frac{e^{-\frac{1}{2} i (2\alpha+2\beta-2\gamma-2\delta+\phi)} \text{Sin}[\frac{\theta}{2}] (2\mathcal{F}_1+2\mathcal{F}_2+(1+\text{Cos}[\theta])(\mathcal{F}_3+\mathcal{F}_4))}{\sqrt{2}}\right\}\right\}$$

Phase relationships.

```
{T0⟦2, 2⟧ / T0⟦1, 1⟧, T0⟦2, 1⟧ / T0⟦1, 2⟧} // Simplify
```

$$\{-e^{-i(\beta-\delta-\phi)}, e^{-i(\beta+\delta+\phi)}\}$$

```
{T-1⟦1, 1⟧ / T1⟦2, 2⟧, T-1⟦1, 2⟧ / T1⟦2, 1⟧,
  T-1⟦2, 1⟧ / T1⟦1, 2⟧, T-1⟦2, 2⟧ / T1⟦1, 1⟧} // Simplify
```

$$\{e^{i(\beta-\delta-3\phi)}, -e^{i(\beta+\delta-\phi)}, -e^{-i(\beta+\delta+3\phi)}, e^{-i(\beta-\delta+\phi)}\}$$

■ Transition matrices using Jacob-Wick phases

```
T0 /. phases[JW] // Simplify
```

$$\left\{\left\{\frac{Q\,\text{Cos}[\frac{\theta}{2}] (\mathcal{F}_5+\mathcal{F}_6)}{\omega}, \frac{e^{i\phi} Q\,\text{Sin}[\frac{\theta}{2}] (\mathcal{F}_5-\mathcal{F}_6)}{\omega}\right\}, \left\{\frac{e^{-i\phi} Q\,\text{Sin}[\frac{\theta}{2}] (\mathcal{F}_5-\mathcal{F}_6)}{\omega}, -\frac{Q\,\text{Cos}[\frac{\theta}{2}] (\mathcal{F}_5+\mathcal{F}_6)}{\omega}\right\}\right\}$$

```
T1 /. phases[JW] // Simplify
```

$$\left\{\left\{\frac{e^{i\phi} \text{Sin}[\frac{\theta}{2}] (2\mathcal{F}_1+2\mathcal{F}_2+(1+\text{Cos}[\theta])(\mathcal{F}_3+\mathcal{F}_4))}{\sqrt{2}}, \frac{e^{2i\phi} \text{Sin}[\frac{\theta}{2}] \text{Sin}[\theta] (\mathcal{F}_3-\mathcal{F}_4)}{\sqrt{2}}\right\}, \left\{-\frac{\text{Cos}[\frac{\theta}{2}] (2\mathcal{F}_1-2\mathcal{F}_2+(-1+\text{Cos}[\theta])(\mathcal{F}_3-\mathcal{F}_4))}{\sqrt{2}}, -\frac{e^{i\phi} (1+\text{Cos}[\theta]) \text{Sin}[\frac{\theta}{2}] (\mathcal{F}_3+\mathcal{F}_4)}{\sqrt{2}}\right\}\right\}$$

```
T₋₁ /. phases[JW] // Simplify
```

$$\left\{\left\{-\frac{e^{-i\phi}\,(1+\text{Cos}[\theta])\,\text{Sin}[\frac{\theta}{2}]\,(\mathcal{F}_3+\mathcal{F}_4)}{\sqrt{2}},\ \frac{\text{Cos}[\frac{\theta}{2}]\,(2\mathcal{F}_1-2\mathcal{F}_2+(-1+\text{Cos}[\theta])\,(\mathcal{F}_3-\mathcal{F}_4))}{\sqrt{2}}\right\},\ \left\{\frac{e^{-2i\phi}\,\text{Cos}[\frac{\theta}{2}]\,(-1+\text{Cos}[\theta])\,(\mathcal{F}_3-\mathcal{F}_4)}{\sqrt{2}},\ \frac{e^{-i\phi}\,\text{Sin}[\frac{\theta}{2}]\,(2\mathcal{F}_1+2\mathcal{F}_2+(1+\text{Cos}[\theta])\,(\mathcal{F}_3+\mathcal{F}_4))}{\sqrt{2}}\right\}\right\}$$

Phase relationships.

```
T₀ /. phases[JW] /. φ → 0 // Simplify
```

$$\left\{\left\{\frac{Q\,\text{Cos}[\frac{\theta}{2}]\,(\mathcal{F}_5+\mathcal{F}_6)}{\omega},\ \frac{Q\,\text{Sin}[\frac{\theta}{2}]\,(\mathcal{F}_5-\mathcal{F}_6)}{\omega}\right\},\ \left\{\frac{Q\,\text{Sin}[\frac{\theta}{2}]\,(\mathcal{F}_5-\mathcal{F}_6)}{\omega},\ -\frac{Q\,\text{Cos}[\frac{\theta}{2}]\,(\mathcal{F}_5+\mathcal{F}_6)}{\omega}\right\}\right\}$$

```
T₁ /. phases[JW] /. φ → 0 // Simplify
```

$$\left\{\left\{\frac{\text{Sin}[\frac{\theta}{2}]\,(2\mathcal{F}_1+2\mathcal{F}_2+(1+\text{Cos}[\theta])\,(\mathcal{F}_3+\mathcal{F}_4))}{\sqrt{2}},\ \frac{\text{Sin}[\frac{\theta}{2}]\,\text{Sin}[\theta]\,(\mathcal{F}_3-\mathcal{F}_4)}{\sqrt{2}}\right\},\ \left\{-\frac{\text{Cos}[\frac{\theta}{2}]\,(2\mathcal{F}_1-2\mathcal{F}_2+(-1+\text{Cos}[\theta])\,(\mathcal{F}_3-\mathcal{F}_4))}{\sqrt{2}},\ -\frac{(1+\text{Cos}[\theta])\,\text{Sin}[\frac{\theta}{2}]\,(\mathcal{F}_3+\mathcal{F}_4)}{\sqrt{2}}\right\}\right\}$$

```
T₋₁ /. phases[JW] /. φ → 0 // Simplify
```

$$\left\{\left\{-\frac{(1+\text{Cos}[\theta])\,\text{Sin}[\frac{\theta}{2}]\,(\mathcal{F}_3+\mathcal{F}_4)}{\sqrt{2}},\ \frac{\text{Cos}[\frac{\theta}{2}]\,(2\mathcal{F}_1-2\mathcal{F}_2+(-1+\text{Cos}[\theta])\,(\mathcal{F}_3-\mathcal{F}_4))}{\sqrt{2}}\right\},\ \left\{\frac{\text{Cos}[\frac{\theta}{2}]\,(-1+\text{Cos}[\theta])\,(\mathcal{F}_3-\mathcal{F}_4)}{\sqrt{2}},\ \frac{\text{Sin}[\frac{\theta}{2}]\,(2\mathcal{F}_1+2\mathcal{F}_2+(1+\text{Cos}[\theta])\,(\mathcal{F}_3+\mathcal{F}_4))}{\sqrt{2}}\right\}\right\}$$

```
{T₀[[2, 2]] / T₀[[1, 1]], T₀[[2, 1]] / T₀[[1, 2]]} /. phases[JW] /. φ → 0 // Simplify
```

```
{-1, 1}
```

```
{T-1⟦1, 1⟧ / T1⟦2, 2⟧, T-1⟦1, 2⟧ / T1⟦2, 1⟧, T-1⟦2, 1⟧ / T1⟦1, 2⟧,
    T-1⟦2, 2⟧ / T1⟦1, 1⟧} /. phases[JW] /. ϕ → 0 // Simplify
```

{1, -1, -1, 1}

Relationship between helicity and CGLN amplitudes

The six independent helicity amplitudes are historically numbered as follows.

```
HelicityToCGLN =
{H1 → T1⟦2, 2⟧, H2 → T1⟦2, 1⟧, H3 → T1⟦1, 2⟧,
    H4 → T1⟦1, 1⟧, H5 → T0⟦1, 1⟧, H6 → T0⟦1, 2⟧} // Simplify
```

$$\Big\{H_1 \to -\frac{e^{-\frac{1}{2} i (2\alpha+2\beta-2\gamma-2\delta-3\phi)} (1+\text{Cos}[\theta])\,\text{Sin}[\frac{\theta}{2}]\,(\mathcal{F}_3+\mathcal{F}_4)}{\sqrt{2}},$$

$$H_2 \to \frac{e^{-\frac{1}{2} i (2\alpha+2\beta-2\gamma-\phi)}\,\text{Cos}[\frac{\theta}{2}]\,(2\mathcal{F}_1-2\mathcal{F}_2+(-1+\text{Cos}[\theta])(\mathcal{F}_3-\mathcal{F}_4))}{\sqrt{2}},$$

$$H_3 \to \frac{e^{-\frac{1}{2} i (2\alpha-2\gamma-2\delta-3\phi)}\,\text{Sin}[\frac{\theta}{2}]\,\text{Sin}[\theta]\,(-\mathcal{F}_3+\mathcal{F}_4)}{\sqrt{2}},$$

$$H_4 \to \frac{e^{-\frac{1}{2} i (2\alpha-2\gamma-\phi)}\,\text{Sin}[\frac{\theta}{2}]\,(2\mathcal{F}_1+2\mathcal{F}_2+(1+\text{Cos}[\theta])(\mathcal{F}_3+\mathcal{F}_4))}{\sqrt{2}},$$

$$H_5 \to \frac{e^{-\frac{1}{2} i (2\alpha-2\gamma+\phi)}\,Q\,\text{Cos}[\frac{\theta}{2}]\,(\mathcal{F}_5+\mathcal{F}_6)}{\omega},$$

$$H_6 \to \frac{e^{-\frac{1}{2} i (2\alpha-2\gamma-2\delta-\phi)}\,Q\,\text{Sin}[\frac{\theta}{2}]\,(-\mathcal{F}_5+\mathcal{F}_6)}{\omega}\Big\}$$

```
HelicityToCGLN /. phases[JW] // Simplify
```

$$\Big\{H_1 \to -\frac{e^{i\phi}(1+\text{Cos}[\theta])\,\text{Sin}[\frac{\theta}{2}]\,(\mathcal{F}_3+\mathcal{F}_4)}{\sqrt{2}},$$

$$H_2 \to -\frac{\text{Cos}[\frac{\theta}{2}]\,(2\mathcal{F}_1-2\mathcal{F}_2+(-1+\text{Cos}[\theta])(\mathcal{F}_3-\mathcal{F}_4))}{\sqrt{2}},$$

$$H_3 \to \frac{e^{2i\phi}\,\text{Sin}[\frac{\theta}{2}]\,\text{Sin}[\theta]\,(\mathcal{F}_3-\mathcal{F}_4)}{\sqrt{2}},$$

$$H_4 \to \frac{e^{i\phi}\,\text{Sin}[\frac{\theta}{2}]\,(2\mathcal{F}_1+2\mathcal{F}_2+(1+\text{Cos}[\theta])(\mathcal{F}_3+\mathcal{F}_4))}{\sqrt{2}},$$

$$H_5 \to \frac{Q\,\text{Cos}[\frac{\theta}{2}]\,(\mathcal{F}_5+\mathcal{F}_6)}{\omega},\ H_6 \to \frac{e^{i\phi}\,Q\,\text{Sin}[\frac{\theta}{2}]\,(\mathcal{F}_5-\mathcal{F}_6)}{\omega}\Big\}$$

It is useful to express the transition matrices in terms of simplified helicity amplitudes whose azimuthal dependencies have been extracted.

```
TH₁ = {{E^(I φ) H₄, E^(2 I φ) H₃}, {H₂, E^(I φ) H₁}};
TH₋₁ = {{E^(-I φ) H₁, -H₂}, {-E^(-2 I φ) H₃, E^(-I φ) H₄}};
TH₀ = {{H₅, E^(I φ) H₆}, {E^(-I φ) H₆, -H₅}};
```

Express response tensor in terms of response functions

In separate notes, I show that the differential cross section for virtual photoexcitation can be expressed in the form

$$\frac{d\sigma}{d\Omega_N} = \frac{p}{K_\gamma}\left[\epsilon\,\mathcal{W}_L + \mathcal{W}_T + \sqrt{\epsilon(1+\epsilon)}\;\mathcal{W}_{LT} + \epsilon\,\mathcal{W}_{TT} + h\sqrt{1-\epsilon^2}\;\mathcal{W}_{TTh} + h\sqrt{\epsilon(1-\epsilon)}\;\mathcal{W}_{LTh}\right]$$

where p is the momentum, $K_\gamma = \frac{W^2 - m_N{}^2}{2W}$ is the equivalent real photon energy (cm), ϵ is the polarization of the virtual photon, and h is the electron helicity. The ϕ-dependent response functions $\mathcal{W}_i$ are given by the functions defined below.

```
𝒲[T_, λ1_, λ2_] :=
  ((-1)^(λ1+λ2) Sum[(HermitianConjugate[Tλ2 /. A_a_ → Aa*].ρf.Tλ1.ρi /. phases[
        JW])⟦k, k⟧, {k, 1, 2}] // Expand) //. ContractAmplitudeProducts

𝒲L[T_] := 2 𝒲[T, 0, 0];
𝒲T[T_] := 𝒲[T, 1, 1] + 𝒲[T, -1, -1];
𝒲LT[T_] := 𝒲[T, 0, 1] - 𝒲[T, 0, -1] + 𝒲[T, 1, 0] - 𝒲[T, -1, 0];
𝒲LTh[T_] := 𝒲[T, 0, 1] + 𝒲[T, 0, -1] + 𝒲[T, 1, 0] + 𝒲[T, -1, 0];
𝒲TT[T_] := -(𝒲[T, 1, -1] + 𝒲[T, -1, 1]);
𝒲TTh[T_] := 𝒲[T, 1, 1] - 𝒲[T, -1, -1];
```

It is well-known that if the polarization vectors are expressed in a basis with $\hat{N}$ normal to the reaction plane, the azimuthal dependence of the observables can be extracted from the response functions. Both the ejectile and target bases defined above have this property, as would any other basis related to these by a rotation about the normal to the reaction plane. However, the precise relationship between the ϕ-dependent response functions and the more common ϕ-independent response functions, R_i, depends upon the conventions chosen by a particular author for normalizations and the choice between longitudinal and scalar amplitudes — unfortunately, many such conventions are found in the literature. We have chosen to express the cross section in the form

$$\begin{aligned}\frac{d\sigma}{d\Omega_N} = \frac{p}{K_\gamma}\Big[&\epsilon_S\,(R_L[0] + \mathcal{P}_N\,R_L[N]) + (R_T[0] + \mathcal{P}_N\,R_T[N]) \\ &+ \sqrt{2\,\epsilon_S\,(1+\epsilon)}\;(R_{LT}[0] + \mathcal{P}_N\,R_{LT}[N])\,\mathrm{Cos}[\phi] + \\ &\sqrt{2\,\epsilon_S\,(1+\epsilon)}\;(\mathcal{P}_L\,R_{LT}[L] + \mathcal{P}_S\,R_{LT}[S])\,\mathrm{Sin}[\phi] \\ &+ \epsilon\,(R_{TT}[0] + \mathcal{P}_N\,R_{TT}[N])\,\mathrm{Cos}[2\,\phi] + \sqrt{\epsilon}\;(\mathcal{P}_L\,R_{TT}[L] + \mathcal{P}_S\,R_{TT}[S])\,\mathrm{Sin}[2\,\phi] \\ &+ h\sqrt{2\,\epsilon_S\,(1-\epsilon)}\;(R_{LTh}[0] + \mathcal{P}_N\,R_{LTh}[N])\,\mathrm{Sin}[\phi] + \\ &h\sqrt{2\,\epsilon_S\,(1-\epsilon)}\;(\mathcal{P}_L\,R_{LTh}[L] + \mathcal{P}_S\,R_{LTh}[S])\,\mathrm{Cos}[\phi] \\ &+ h\sqrt{1-\epsilon^2}\;(\mathcal{P}_L\,R_{TTh}[L] + \mathcal{P}_S\,R_{TTh}[S])\Big]\end{aligned}$$

where $\epsilon_S = \frac{Q^2}{q^2}\,\epsilon$ in the barycentric frame and where $\{\mathcal{P}_S, \mathcal{P}_N, \mathcal{P}_L\}$ are components of the nucleon polarization. In this representation it is natural to express the R_L and R_{LT} response functions in terms of scalar multipole amplitudes. Other authors replace ϵ_S by $\epsilon_L = \frac{Q^2}{\omega^2}\epsilon$ and employ longitudinal instead of scalar multipoles. However, additional differences in signs and factors of $\sqrt{2}$ are common also.

Recoil-Polarization Response Functions

■ Recoil polarization vector, projection operator, and density matrix

```
P̂ = P_L L̂ + P_N N̂ + P_S Ŝ /. EjectileBasis
```

$\{-\text{Cos}[\phi]\,\text{Sin}[\theta]\,P_L + \text{Sin}[\phi]\,P_N + \text{Cos}[\theta]\,\text{Cos}[\phi]\,P_S,$
$-\text{Sin}[\theta]\,\text{Sin}[\phi]\,P_L - \text{Cos}[\phi]\,P_N + \text{Cos}[\theta]\,\text{Sin}[\phi]\,P_S,\ -\text{Cos}[\theta]\,P_L - \text{Sin}[\theta]\,P_S\}$

$\mathbb{P} = \frac{1}{2}\left(\text{IdentityMatrix[2]} + \hat{P}.\vec{\sigma}\right)$ /. trigToExp[ϕ] // Simplify;

$\mathbb{P}$ // MatrixForm

$$\begin{pmatrix} \frac{1}{2}(1 - \text{Cos}[\theta]\,P_L - \text{Sin}[\theta]\,P_S) & \frac{1}{2}e^{-i\phi}(-\text{Sin}[\theta]\,P_L + i\,P_N + \text{Cos}[\theta]\,P_S) \\ -\frac{1}{2}e^{i\phi}(\text{Sin}[\theta]\,P_L + i\,P_N - \text{Cos}[\theta]\,P_S) & \frac{1}{2}(1 + \text{Cos}[\theta]\,P_L + \text{Sin}[\theta]\,P_S) \end{pmatrix}$$

$\rho_i = \frac{1}{2}$ IdentityMatrix[2];
ρ_f = HermitianConjugate[χ_f].$\mathbb{P}$.χ_f // Simplify;

ρ_f // MatrixForm

$$\begin{pmatrix} \frac{1}{2}(1 + P_L) & \frac{1}{2}\,i\,e^{i\beta}(P_N + i\,P_S) \\ -\frac{1}{2}\,i\,e^{-i\beta}(P_N - i\,P_S) & \frac{1}{2}(1 - P_L) \end{pmatrix}$$

ρ_f /. phases[JW] // Simplify // MatrixForm

$$\begin{pmatrix} \frac{1}{2}(1 + P_L) & \frac{1}{2}e^{i\phi}(-i\,P_N + P_S) \\ \frac{1}{2}e^{-i\phi}(i\,P_N + P_S) & \frac{1}{2}(1 - P_L) \end{pmatrix}$$

■ ϕ-dependent response functions in terms of helicity amplitudes

($\mathcal{W}_L$[TH] // Simplify) // Collect[#, {$P_{x\_}$, f_[a_. ϕ]}] &

$\text{Abs}[H_5]^2 + \text{Abs}[H_6]^2 - 2\,\text{Im}[H_5\,(H_6)^*]\,P_N$

```
(𝒲T[TH] // Simplify) // Collect[#, {Px_, f_[a_. ϕ]}] &
```

$$\frac{1}{2}\left(\text{Abs}[H_1]^2+\text{Abs}[H_2]^2+\text{Abs}[H_3]^2+\text{Abs}[H_4]^2\right)+\frac{1}{2}\left(2\,\text{Im}[H_1 (H_3)^*]+2\,\text{Im}[H_2 (H_4)^*]\right) P_N$$

```
(𝒲LT[TH] // ExpToTrig // MyFullSimplify) //.
  ContractAmplitudeProducts // Collect[#, {Px_, f_[a_. ϕ]}] &
```

$$\text{Cos}[\phi]\left(\text{Re}[H_1 (H_5)^*]-\text{Re}[H_4 (H_5)^*]-\text{Re}[(H_2+H_3)(H_6)^*]\right)+\left(\text{Im}[(H_1+H_4)(H_5)^*]-\text{Im}[(H_2-H_3)(H_6)^*]\right)\text{Sin}[\phi]\,P_L+\text{Cos}[\phi]\left(-\text{Im}[(H_2+H_3)(H_5)^*]-\text{Im}[(H_1-H_4)(H_6)^*]\right)P_N+\left(\text{Im}[(H_2-H_3)(H_5)^*]+\text{Im}[(H_1+H_4)(H_6)^*]\right)\text{Sin}[\phi]\,P_S$$

```
(𝒲LTh[TH] // ExpToTrig // MyFullSimplify) //.
  ContractAmplitudeProducts // Collect[#, {Px_, f_[a_. ϕ]}] &
```

$$\left(-\text{Im}[(H_1-H_4)(H_5)^*]+\text{Im}[(H_2+H_3)(H_6)^*]\right)\text{Sin}[\phi]+\text{Cos}[\phi]\left(-\text{Re}[(H_1+H_4)(H_5)^*]+\text{Re}[(H_2-H_3)(H_6)^*]\right)P_L+\left(-\text{Re}[(H_2+H_3)(H_5)^*]-\text{Re}[(H_1-H_4)(H_6)^*]\right)\text{Sin}[\phi]\,P_N+\text{Cos}[\phi]\left(-\text{Re}[(H_2-H_3)(H_5)^*]-\text{Re}[(H_1+H_4)(H_6)^*]\right)P_S$$

```
(𝒲TT[TH] // ExpToTrig // MyFullSimplify) //.
  ContractAmplitudeProducts // Collect[#, {Px_, f_[a_. ϕ]}] &
```

$$\text{Cos}[2\phi]\left(\text{Re}[H_2 (H_3)^*]-\text{Re}[H_1 (H_4)^*]\right)+\left(\text{Im}[H_2 (H_3)^*]-\text{Im}[H_1 (H_4)^*]\right)\text{Sin}[2\phi]\,P_L+\text{Cos}[2\phi]\left(\text{Im}[H_1 (H_2)^*]+\text{Im}[H_3 (H_4)^*]\right)P_N+\left(-\text{Im}[H_1 (H_2)^*]+\text{Im}[H_3 (H_4)^*]\right)\text{Sin}[2\phi]\,P_S$$

```
(𝒲TTh[TH] // MyFullSimplify) //. ContractAmplitudeProducts //
 Collect[#, {Px_, f_[a_. ϕ]}] &
```

$$\frac{1}{2}\left(-\text{Abs}[H_1]^2-\text{Abs}[H_2]^2+\text{Abs}[H_3]^2+\text{Abs}[H_4]^2\right)P_L+\left(\text{Re}[H_1 (H_3)^*]+\text{Re}[H_2 (H_4)^*]\right)P_S$$

■ ϕ-independent response functions in terms of helicity amplitudes

```
tempRL = 𝒲L[TH] // MySimplify // Collect[#, {Px_, f_[a_. ϕ]}] &;
RL[H] = {RL[0] → Select[tempRL, FreeQ[#, Px_] &],
   RL[N] → Coefficient[tempRL, PN] } // Simplify
```

$$\{R_L[0]\to\text{Abs}[H_5]^2+\text{Abs}[H_6]^2,\ R_L[N]\to -2\,\text{Im}[H_5 (H_6)^*]\}$$

```
tempR_T = (𝒲_T[TH] // MySimplify //. ContractAmplitudeProducts //
    Collect[#, {P_x_, f_[a_. ϕ]}] &);
R_T[H] = {R_T[0] → Select[tempR_T, FreeQ[#, P_x_] &],
   R_T[N] → Coefficient[tempR_T, P_N] } // Simplify
```

$\{R_T[0] \to \frac{1}{2}(\text{Abs}[H_1]^2 + \text{Abs}[H_2]^2 + \text{Abs}[H_3]^2 + \text{Abs}[H_4]^2),$
$R_T[N] \to \text{Im}[H_1 (H_3)^*] + \text{Im}[H_2 (H_4)^*]\}$

```
tempR_LT = (𝒲_LT[TH] // ExpToTrig // MySimplify) //.
    ContractAmplitudeProducts // Collect[#, {P_x_, f_[a_. ϕ]}] &;
R_LT[H] = {
    R_LT[0] → Select[tempR_LT, FreeQ[#, P_x_] &] / Cos[ϕ],
    R_LT[N] → Coefficient[tempR_LT, P_N] / Cos[ϕ],
    R_LT[L] → Coefficient[tempR_LT, P_L] / Sin[ϕ],
    R_LT[S] → Coefficient[tempR_LT, P_S] / Sin[ϕ]} // Simplify
```

$\{R_{LT}[0] \to \text{Re}[(H_1 - H_4)(H_5)^*] - \text{Re}[(H_2 + H_3)(H_6)^*],$
$R_{LT}[N] \to -\text{Im}[(H_2 + H_3)(H_5)^*] - \text{Im}[(H_1 - H_4)(H_6)^*],$
$R_{LT}[L] \to \text{Im}[(H_1 + H_4)(H_5)^*] - \text{Im}[(H_2 - H_3)(H_6)^*],$
$R_{LT}[S] \to \text{Im}[(H_2 - H_3)(H_5)^*] + \text{Im}[(H_1 + H_4)(H_6)^*]\}$

```
tempR_LTh =
  (𝒲_LTh[TH] // ExpToTrig // MySimplify) //. ContractAmplitudeProducts //
   Collect[#, {P_x_, f_[a_. ϕ], f_[a_. θ]}] &;
R_LTh[H] = {
    R_LTh[0] → Select[tempR_LTh, FreeQ[#, P_x_] &] / Sin[ϕ],
    R_LTh[N] → Coefficient[tempR_LTh, P_N] / Sin[ϕ],
    R_LTh[L] → Coefficient[tempR_LTh, P_L] / Cos[ϕ],
    R_LTh[S] → Coefficient[tempR_LTh, P_S] / Cos[ϕ]} // Simplify
```

$\{R_{LTh}[0] \to -\text{Im}[(H_1 - H_4)(H_5)^*] + \text{Im}[(H_2 + H_3)(H_6)^*],$
$R_{LTh}[N] \to -\text{Re}[(H_2 + H_3)(H_5)^*] - \text{Re}[(H_1 - H_4)(H_6)^*],$
$R_{LTh}[L] \to -\text{Re}[(H_1 + H_4)(H_5)^*] + \text{Re}[(H_2 - H_3)(H_6)^*],$
$R_{LTh}[S] \to -\text{Re}[(H_2 - H_3)(H_5)^*] - \text{Re}[(H_1 + H_4)(H_6)^*]\}$

```
tempR_TT =
  (𝒲_TT[TH] // ExpToTrig // MySimplify) // Collect[#, {P_x_, f_[a_. ϕ]}] &;
R_TT[H] = {
    R_TT[0] → Select[tempR_TT, FreeQ[#, P_x_] &] / Cos[2 ϕ],
    R_TT[N] → Coefficient[tempR_TT, P_N] / Cos[2 ϕ],
    R_TT[L] → Coefficient[tempR_TT, P_L] / Sin[2 ϕ],
    R_TT[S] → Coefficient[tempR_TT, P_S] / Sin[2 ϕ]} //
  Simplify //. ContractAmplitudeProducts
```

$\{R_{TT}[0] \to \text{Re}[H_2 (H_3)^*] - \text{Re}[H_1 (H_4)^*],\ R_{TT}[N] \to \text{Im}[H_1 (H_2)^*] + \text{Im}[H_3 (H_4)^*],$
$R_{TT}[L] \to \text{Im}[H_2 (H_3)^*] - \text{Im}[H_1 (H_4)^*],\ R_{TT}[S] \to -\text{Im}[H_1 (H_2)^*] + \text{Im}[H_3 (H_4)^*]\}$

```
tempR_TTh =
  (𝒲_TTh[TH] // ExpToTrig // MySimplify) // Collect[#, {P_x_, f_[a_. ϕ]}] &;
R_TTh[H] = {
    R_TTh[0] → Select[tempR_TTh, FreeQ[#, P_x_] &],
    R_TTh[N] → Coefficient[tempR_TTh, P_N],
    R_TTh[L] → Coefficient[tempR_TTh, P_L],
    R_TTh[S] → Coefficient[tempR_TTh, P_S]} // Simplify
```

$\{R_{TTh}[0] \to 0,\ R_{TTh}[N] \to 0,$

$R_{TTh}[L] \to \frac{1}{2}\ (-\text{Abs}[H_1]^2 - \text{Abs}[H_2]^2 + \text{Abs}[H_3]^2 + \text{Abs}[H_4]^2),$

$R_{TTh}[S] \to \text{Re}[H_1\ (H_3)^*] + \text{Re}[H_2\ (H_4)^*]\}$

■ ϕ-independent response functions in terms of CGLN amplitudes

```
tempR_L = 𝒲_L[T] // Simplify // Collect[#, {P_x_, f_[a_. ϕ]}] &;
R_L[ℱ] = {
    R_L[0] → Select[tempR_L, FreeQ[#, P_x_] &],
    R_L[N] → Coefficient[tempR_L, P_N] } // FullSimplify
```

$\{R_L[0] \to \frac{Q^2\ (\text{Abs}[\mathcal{F}_5]^2 + \text{Abs}[\mathcal{F}_6]^2 + 2\ \text{Cos}[\theta]\ \text{Re}[\mathcal{F}_5\ (\mathcal{F}_6)^*])}{\omega^2},$

$R_L[N] \to \frac{2\ Q^2\ \text{Im}[\mathcal{F}_5\ (\mathcal{F}_6)^*]\ \text{Sin}[\theta]}{\omega^2}\}$

```
tempR_T = (𝒲_T[T] // Simplify // Collect[#, {P_x_, f_[a_. ϕ]}] &);
R_T[ℱ] = {
    R_T[0] → Select[tempR_T, FreeQ[#, P_x_] &],
    R_T[N] → Coefficient[tempR_T, P_N] } // FullSimplify
```

$\{R_T[0] \to$

$\text{Abs}[\mathcal{F}_1]^2 + \text{Abs}[\mathcal{F}_2]^2 - 2\ \text{Cos}[\theta]\ \text{Re}[\mathcal{F}_1\ (\mathcal{F}_2)^*] + \frac{1}{2}\ (\text{Abs}[\mathcal{F}_3]^2 + \text{Abs}[\mathcal{F}_4]^2 +$

$2\ (\text{Re}[\mathcal{F}_2\ (\mathcal{F}_3)^*] + \text{Re}[\mathcal{F}_1\ (\mathcal{F}_4)^*] + \text{Cos}[\theta]\ \text{Re}[\mathcal{F}_3\ (\mathcal{F}_4)^*]))\ \text{Sin}[\theta]^2,$

$R_T[N] \to \text{Sin}[\theta]\ (-2\ \text{Im}[\mathcal{F}_1\ (\mathcal{F}_2)^*] - \text{Im}[\mathcal{F}_1\ (\mathcal{F}_3)^*] + \text{Cos}[\theta]$

$(\text{Im}[\mathcal{F}_2\ (\mathcal{F}_3)^*] - \text{Im}[\mathcal{F}_1\ (\mathcal{F}_4)^*]) + \text{Im}[\mathcal{F}_2\ (\mathcal{F}_4)^*] + \text{Im}[\mathcal{F}_3\ (\mathcal{F}_4)^*]\ \text{Sin}[\theta]^2)\}$

```
tempR_LT = 𝒲_LT[T] // ExpToTrig // MySimplify //
   Collect[#, {P_x_, f_[a_. ϕ]}] & //. ContractAmplitudeProducts;
R_LT[ℱ] = {
    R_LT[0] → Select[tempR_LT, FreeQ[#, P_x_] &] / Cos[ϕ],
    R_LT[N] → Coefficient[tempR_LT, P_N] / Cos[ϕ],
    R_LT[L] → Coefficient[tempR_LT, P_L] / Sin[ϕ],
    R_LT[S] → Coefficient[tempR_LT, P_S] / Sin[ϕ]} // FullSimplify
```

$$\left\{R_{LT}[0] \to -\frac{1}{\omega}\left(\sqrt{2}\, Q\, (\mathrm{Re}[\mathcal{F}_2\, (\mathcal{F}_5)^*] + \mathrm{Re}[\mathcal{F}_3\, (\mathcal{F}_5)^*] + \mathrm{Re}[\mathcal{F}_1\, (\mathcal{F}_6)^*] + \mathrm{Cos}[\theta]\, (\mathrm{Re}[\mathcal{F}_4\, (\mathcal{F}_5)^*] + \mathrm{Re}[\mathcal{F}_3\, (\mathcal{F}_6)^*]) + \mathrm{Re}[\mathcal{F}_4\, (\mathcal{F}_6)^*])\, \mathrm{Sin}[\theta]\right),\right.$$

$$R_{LT}[N] \to \frac{1}{\omega}\left(\sqrt{2}\, Q\, (\mathrm{Im}[\mathcal{F}_1\, (\mathcal{F}_5)^*] + \mathrm{Cos}[\theta]\, (-\mathrm{Im}[\mathcal{F}_2\, (\mathcal{F}_5)^*] + \mathrm{Im}[\mathcal{F}_1\, (\mathcal{F}_6)^*]) - \mathrm{Im}[\mathcal{F}_2\, (\mathcal{F}_6)^*] + (\mathrm{Im}[\mathcal{F}_4\, (\mathcal{F}_5)^*] - \mathrm{Im}[\mathcal{F}_3\, (\mathcal{F}_6)^*])\, \mathrm{Sin}[\theta]^2)\right),$$

$$R_{LT}[L] \to \frac{\sqrt{2}\, Q\, (\mathrm{Im}[\mathcal{F}_1\, (\mathcal{F}_5)^*] + \mathrm{Im}[\mathcal{F}_2\, (\mathcal{F}_6)^*])\, \mathrm{Sin}[\theta]}{\omega},$$

$$R_{LT}[S] \to \frac{\sqrt{2}\, Q\, (\mathrm{Im}[\mathcal{F}_2\, (\mathcal{F}_5)^*] - \mathrm{Im}[\mathcal{F}_1\, (\mathcal{F}_6)^*] + \mathrm{Cos}[\theta]\, (-\mathrm{Im}[\mathcal{F}_1\, (\mathcal{F}_5)^*] + \mathrm{Im}[\mathcal{F}_2\, (\mathcal{F}_6)^*]))}{\omega}$$

$$\}$$

```
tempR_LTh = 𝒲_LTh[T] // ExpToTrig // MySimplify //
   Collect[#, {P_x_, f_[a_. ϕ], f_[a_. θ]}] &;
R_LTh[ℱ] = {
    R_LTh[0] → Select[tempR_LTh, FreeQ[#, P_x_] &] / Sin[ϕ],
    R_LTh[N] → Coefficient[tempR_LTh, P_N] / Sin[ϕ],
    R_LTh[L] → Coefficient[tempR_LTh, P_L] / Cos[ϕ],
    R_LTh[S] → Coefficient[tempR_LTh, P_S] / Cos[ϕ]} // FullSimplify
```

$$\left\{R_{LTh}[0] \to \frac{1}{\omega}\left(\sqrt{2}\, Q\, (\mathrm{Im}[\mathcal{F}_2\, (\mathcal{F}_5)^*] + \mathrm{Im}[\mathcal{F}_3\, (\mathcal{F}_5)^*] + \mathrm{Im}[\mathcal{F}_1\, (\mathcal{F}_6)^*] + \mathrm{Cos}[\theta]\, (\mathrm{Im}[\mathcal{F}_4\, (\mathcal{F}_5)^*] + \mathrm{Im}[\mathcal{F}_3\, (\mathcal{F}_6)^*]) + \mathrm{Im}[\mathcal{F}_4\, (\mathcal{F}_6)^*])\, \mathrm{Sin}[\theta]\right),\right.$$

$$R_{LTh}[N] \to \frac{1}{\omega}\left(\sqrt{2}\, Q\, (\mathrm{Re}[\mathcal{F}_1\, (\mathcal{F}_5)^*] + \mathrm{Cos}[\theta]\, (-\mathrm{Re}[\mathcal{F}_2\, (\mathcal{F}_5)^*] + \mathrm{Re}[\mathcal{F}_1\, (\mathcal{F}_6)^*]) - \mathrm{Re}[\mathcal{F}_2\, (\mathcal{F}_6)^*] + (\mathrm{Re}[\mathcal{F}_4\, (\mathcal{F}_5)^*] - \mathrm{Re}[\mathcal{F}_3\, (\mathcal{F}_6)^*])\, \mathrm{Sin}[\theta]^2)\right),$$

$$R_{LTh}[L] \to -\frac{\sqrt{2}\, Q\, (\mathrm{Re}[\mathcal{F}_1\, (\mathcal{F}_5)^*] + \mathrm{Re}[\mathcal{F}_2\, (\mathcal{F}_6)^*])\, \mathrm{Sin}[\theta]}{\omega},$$

$$R_{LTh}[S] \to \frac{\sqrt{2}\, Q\, (-\mathrm{Re}[\mathcal{F}_2\, (\mathcal{F}_5)^*] + \mathrm{Re}[\mathcal{F}_1\, (\mathcal{F}_6)^*] + \mathrm{Cos}[\theta]\, (\mathrm{Re}[\mathcal{F}_1\, (\mathcal{F}_5)^*] - \mathrm{Re}[\mathcal{F}_2\, (\mathcal{F}_6)^*]))}{\omega}$$

$$\}$$

```
tempR_TT =
  𝒲_TT[T] // ExpToTrig // MySimplify // Collect[#, {P_x_, f_[a_. ϕ]}] &;
R_TT[ℱ] = {
    R_TT[0] → Select[tempR_TT, FreeQ[#, P_x_] &] / Cos[2 ϕ],
    R_TT[N] → Coefficient[tempR_TT, P_N] / Cos[2 ϕ],
    R_TT[L] → Coefficient[tempR_TT, P_L] / Sin[2 ϕ],
    R_TT[S] → Coefficient[tempR_TT, P_S] / Sin[2 ϕ]} // FullSimplify
```

```
{R_TT[0] → 1/2 (Abs[ℱ3]^2 + Abs[ℱ4]^2 +
      2 (Re[ℱ2 (ℱ3)^*] + Re[ℱ1 (ℱ4)^*] + Cos[θ] Re[ℱ3 (ℱ4)^*])) Sin[θ]^2,
 R_TT[N] → Sin[θ] (-Im[ℱ1 (ℱ3)^*] + Cos[θ] (Im[ℱ2 (ℱ3)^*] - Im[ℱ1 (ℱ4)^*]) +
      Im[ℱ2 (ℱ4)^*] + Im[ℱ3 (ℱ4)^*] Sin[θ]^2),
 R_TT[L] → -(Im[ℱ1 (ℱ3)^*] + Im[ℱ2 (ℱ4)^*]) Sin[θ]^2, R_TT[S] →
  (-Im[ℱ2 (ℱ3)^*] + Im[ℱ1 (ℱ4)^*] + Cos[θ] (Im[ℱ1 (ℱ3)^*] - Im[ℱ2 (ℱ4)^*]))
   Sin[θ]}
```

```
tempR_TTh =
  𝒲_TTh[T] // ExpToTrig // MySimplify // Collect[#, {P_x_, f_[a_. ϕ]}] &;
R_TTh[ℱ] = {
    R_TTh[0] → Select[tempR_TTh, FreeQ[#, P_x_] &],
    R_TTh[N] → Coefficient[tempR_TTh, P_N],
    R_TTh[L] → Coefficient[tempR_TTh, P_L],
    R_TTh[S] → Coefficient[tempR_TTh, P_S]} // FullSimplify
```

```
{R_TTh[0] → 0, R_TTh[N] → 0,
 R_TTh[L] → -(Abs[ℱ1]^2 + Abs[ℱ2]^2) Cos[θ] + 2 Re[ℱ1 (ℱ2)^*] +
    (Re[ℱ1 (ℱ3)^*] + Re[ℱ2 (ℱ4)^*]) Sin[θ]^2,
 R_TTh[S] → (-Abs[ℱ1]^2 + Abs[ℱ2]^2 + Re[ℱ2 (ℱ3)^*] - Re[ℱ1 (ℱ4)^*] +
      Cos[θ] (-Re[ℱ1 (ℱ3)^*] + Re[ℱ2 (ℱ4)^*])) Sin[θ]}
```

Multipole expansions

■ Multipole expansion of CGLN amplitudes

Here we quote the traditional expansions of CGLN amplitudes in terms of multipoles. The expressions were obtained from Dennery and are widely quoted in the literature, but I have not checked them independently. [Actually, I did derive amplitudes 1-4 a long time ago, but my notes are somewhat sketchy and have not been reviewed recently. Nevertheless, I am confident that these expressions are correct.]

```
CGLN1[ℓ_] = {
   ℱ1 → Sum[
    (ℓ M+[ℓ] + E+[ℓ]) P′ℓ+1[x] + ((ℓ + 1) M-[ℓ] + E-[ℓ]) P′ℓ-1[x], {ℓ, 0, ℓmax}],
   ℱ2 → Sum[((ℓ + 1) M+[ℓ] + ℓ M-[ℓ]) P′ℓ[x], {ℓ, 0, ℓmax}],
   ℱ3 →
  Sum[(E+[ℓ] - M+[ℓ]) P″ℓ+1[x] + (E-[ℓ] + M-[ℓ]) P″ℓ-1[x], {ℓ, 0, ℓmax}],
   ℱ4 → Sum[(M+[ℓ] - E+[ℓ] - M-[ℓ] - E-[ℓ]) P″ℓ[x], {ℓ, 0, ℓmax}],
   ℱ5 → ω/q Sum[((ℓ + 1) S+[ℓ] P′ℓ+1[x] - ℓ S-[ℓ] P′ℓ-1[x]), {ℓ, 0, ℓmax}],
   ℱ6 → ω/q Sum[((ℓ S-[ℓ] - (ℓ + 1) S+[ℓ]) P′ℓ[x]), {ℓ, 0, ℓmax}]
      };
```

Note that is simplest to allow the sums to begin with $\ell = 0$ and to eliminate nonphysical amplitudes later.

```
rule[mp] = {M-[0] → 0, E-[0] → 0, S-[0] → 0, M+[0] → 0, E-[1] → 0, 0* → 0};
```

It is useful to combine these rules with expansions for the complex conjugates. We include rules for expanding products of sums also.

```
ExpandAmplitudeProducts = {
   Abs[ℱa_]^2 → ℱa (ℱa)*,
   Re[(A_. ℱa_ + B_. ℱb_) (C_. ℱc_ + D_. ℱd_)*] →
    A C Re[ℱa (ℱc)*] + A D Re[ℱa (ℱd)*] + B C Re[ℱb (ℱc)*] + B D Re[ℱb (ℱd)*],
   Im[(A_. ℱa_ + B_. ℱb_) (C_. ℱc_ + D_. ℱd_)*] →
    A C Im[ℱa (ℱc)*] + A D Im[ℱa (ℱd)*] + B C Im[ℱb (ℱc)*] + B D Im[ℱb (ℱd)*],
   Re[(A_. ℱa_ + B_. ℱb_) (C_. ℱc_)* ] → A C Re[ℱa (ℱc)*] + B C Re[ℱb (ℱc)*],
   Im[(A_. ℱa_ + B_. ℱb_) (C_. ℱc_)* ] → A C Im[ℱa (ℱc)*] + B C Im[ℱb (ℱc)*]};

ExpandCGLN =
  Join[{Sum[A_, {ℓ1_, ℓ1min_, ℓ1max_}] Sum[B_, {ℓ2_, ℓ2min_, ℓ2max_}] :→
     MySum[Expand[A B], {ℓ1, ℓ1min, ℓ1max}, {ℓ2, ℓ2min, ℓ2max}] },
   CGLN1[ℓ1], CGLN1[ℓ2] /. {ℱa_ → (ℱa)*, M+[a_] → (M+[a])*,
     E+[a_] → (E+[a])*, S+[a_] → (S+[a])*, M-[a_] → (M-[a])*,
     E-[a_] → (E-[a])*, S-[a_] → (S-[a])*}];
```

The following rules expand the Legendre polynomials and their derivatives.

```
ExpandLegendre = {Pλ_[x] → LegendreP[λ, x],
  P′λ_[x] → D[LegendreP[λ, x], x], P″λ_[x] → D[LegendreP[λ, x], x, x]};
```

The following rules are designed to expand expressions to the form $\sum a_i f[A B^*]$ where f is *Re*, *Im*, or *Abs* and where amplitudes products $A B^*$ are unique with a prescribed ordering hierarchy.

```
AnyMP = M- | M+ | E- | E+ | S- | S+; AnyMP- = M- | E- | S-; AnyMP+ = M+ | E+ | S+;
```

```
ExpandMultipoleProducts = {
   (a_ (x : AnyMP) [m_])* → a (x[m])*,
   Re[a_. x_ (x_)*] → a Abs[x]^2,
   Im[a_. x_ (x_)*] → 0, Abs[f_[θ]]^2 → f[θ]^2,
   Abs[a_. (b_ + c_)]^2 → Abs[a]^2 (Abs[b]^2 + Abs[c]^2 + 2 Re[b c*]),
   Abs[a_. (x : (AnyMP)) [m_]]^2 → a^2 Abs[x[m]]^2,
   (f : (Re | Im)) [a_. (b_ + c_)] → f[a b] + f[a c],
   (f : (Re | Im)) [a_. ((b_ + c_) d_.)*] → f[a (b d)*] + f[a (c d)*],
   (f : (Re | Im)) [a_ (x : AnyMP) [m_] ((y : AnyMP) [n_])*] →
    a f[(y[n])* x[m]],
   Re[((x : (M₋ | M₊)) [m_])* (y : (E₋ | E₊ | S₋ | S₊)) [n_]] → Re[(y[n])* x[m]],
   Im[((x : (M₋ | M₊)) [m_])* (y : (E₋ | E₊ | S₋ | S₊)) [n_]] →
    -Im[(y[n])* x[m]],
   Re[((x : (E₋ | E₊)) [m_])* (y : (S₋ | S₊)) [n_]] → Re[(y[n])* x[m]],
   Im[((x : (E₋ | E₊)) [m_])* (y : (S₋ | S₊)) [n_]] → -Im[(y[n])* x[m]],
   Re[(x : M₊[m_])* y : M₋[n_]] → Re[y* x],
   Im[(x : M₊[m_])* y : M₋[n_]] → -Im[y* x],
   Re[(x : E₊[m_])* y : E₋[n_]] → Re[y* x],
   Im[(x : E₊[m_])* y : E₋[n_]] → -Im[y* x],
   Re[(x : S₊[m_])* y : S₋[n_]] → Re[y* x],
   Im[(x : S₊[m_])* y : S₋[n_]] → -Im[y* x],
   Re[(x : AnyMP) [m_] ((x : AnyMP) [n_])*] /; (n < m) → Re[x[m]* x[n]],
   Im[(x : AnyMP) [m_] ((x : AnyMP) [n_])*] /; (n < m) → -Im[x[m]* x[n]]
};
```

■ Functions which perform multipole expansion of response functions

To obtain tractable expressions, it is necessary to specify the maximum angular momentum. The following function constructs a multipole expansion for a specific response function and attempts to perform simplification. It is useful to express the expansions in terms of $x = \mathrm{Cos}[\theta_\pi]$ and for some response functions it is helpful to extract factors of either $\mathrm{Sin}[\theta_\pi]$ or $\mathrm{Sin}[\theta_\pi]^2$.

```
ExpandR[Rα_[a_], lmax_Integer /; (lmax ≥ 0), n_Integer : 0] :=
 Module[{ℛ},
     ℛ = TrigExpand[
    (Rα[a] /. Rα[ℱ] //. ExpandCGLN /. {ℓmax → lmax, MySum → Sum}) //.
     ExpandMultipoleProducts //. rule[mp]] /.
   {Cos[θ] → x, Sin[θ] → Sqrt[1 - x^2]} /. ExpandLegendre;
  Collect[MySimplify[Sin[θ]^n ℛ / (1 - x^2)^(n/2)], {Sin[θ], x}, MySimplify]]
```

```
ExpandR[R_T[0], 1]
```

$$\text{Abs}[E_+[0]]^2 + \frac{9}{2}\text{Abs}[E_+[1]]^2 + \text{Abs}[M_-[1]]^2 + \frac{5}{2}\text{Abs}[M_+[1]]^2 + x\,(-2\,\text{Re}[M_-[1]\,E_+[0]^*] + 2\,\text{Re}[M_+[1]\,E_+[0]^*] + 6\,\text{Re}[E_+[0]\,E_+[1]^*]) + 3\,\text{Re}[M_-[1]\,E_+[1]^*] - 3\,\text{Re}[M_+[1]\,E_+[1]^*] + \frac{3}{2}x^2\,(3\,\text{Abs}[E_+[1]]^2 - \text{Abs}[M_+[1]]^2 - 6\,\text{Re}[M_-[1]\,E_+[1]^*] + 6\,\text{Re}[M_+[1]\,E_+[1]^*] - 2\,\text{Re}[M_+[1]\,M_-[1]^*]) + \text{Re}[M_+[1]\,M_-[1]^*]$$

```
Simplify /@ CoefficientList[ExpandR[R_T[0], 1], x]
```

$$\{\text{Abs}[E_+[0]]^2 + \frac{9}{2}\text{Abs}[E_+[1]]^2 + \text{Abs}[M_-[1]]^2 + \frac{5}{2}\text{Abs}[M_+[1]]^2 + 3\,\text{Re}[M_-[1]\,E_+[1]^*] - 3\,\text{Re}[M_+[1]\,E_+[1]^*] + \text{Re}[M_+[1]\,M_-[1]^*],\; -2\,\text{Re}[M_-[1]\,E_+[0]^*] + 2\,\text{Re}[M_+[1]\,E_+[0]^*] + 6\,\text{Re}[E_+[0]\,E_+[1]^*],\; \frac{3}{2}(3\,\text{Abs}[E_+[1]]^2 - \text{Abs}[M_+[1]]^2 - 6\,\text{Re}[M_-[1]\,E_+[1]^*] + 6\,\text{Re}[M_+[1]\,E_+[1]^*] - 2\,\text{Re}[M_+[1]\,M_-[1]^*])\}$$

```
ExpandR[R_TT[0], 1, 2]
```

$$\frac{1}{2}(9\,\text{Abs}[E_+[1]]^2 - 3\,(\text{Abs}[M_+[1]]^2 + 2\,(-\text{Re}[M_-[1]\,E_+[1]^*] + \text{Re}[M_+[1]\,E_+[1]^*] + \text{Re}[M_+[1]\,M_-[1]^*])))\,\text{Sin}[\theta]^2$$

The following function eliminates terms which do not involve particular multipoles.

```
Clear[MyFreeQ];
MyFreeQ[x_, y_List] := And @@ (FreeQ[x, #] & /@ y );
MyFreeQ[x_, y_] := FreeQ[x, y]

AbbreviateMultipoleExpansion[expr_, choices_] :=
 Collect[MySimplify[(If[MyFreeQ[#, choices], 0, #] &) /@ Expand[expr]],
  x, MySimplify]
```

For example, we can enforce $M_+[1]$ dominance for π production near Δ resonance.

```
AbbreviateMultipoleExpansion[ ExpandR[R_T[0], 1], M_+[1]]
```

$$\frac{5}{2}\text{Abs}[M_+[1]]^2 + 2\,x\,\text{Re}[M_+[1]\,E_+[0]^*] - 3\,\text{Re}[M_+[1]\,E_+[1]^*] + \text{Re}[M_+[1]\,M_-[1]^*] - \frac{3}{2}x^2\,(\text{Abs}[M_+[1]]^2 - 6\,\text{Re}[M_+[1]\,E_+[1]^*] + 2\,\text{Re}[M_+[1]\,M_-[1]^*])$$

■ Results for selected response functions

■ Expansions through *s* and *p* waves

ExpandR[R_L[0], 1]

$$\frac{2\,Q^2\,x\,(\text{Re}[S_-[1]^*\,S_+[0]] + 4\,\text{Re}[S_+[1]^*\,S_+[0]])}{q^2} + \frac{Q^2\,(\text{Abs}[S_-[1]]^2 + \text{Abs}[S_+[0]]^2 + 4\,\text{Abs}[S_+[1]]^2 - 4\,\text{Re}[S_-[1]^*\,S_+[1]])}{q^2} + \frac{12\,Q^2\,x^2\,(\text{Abs}[S_+[1]]^2 + \text{Re}[S_-[1]^*\,S_+[1]])}{q^2}$$

ExpandR[R_T[0], 1]

$$\text{Abs}[E_+[0]]^2 + \frac{9}{2}\,\text{Abs}[E_+[1]]^2 + \text{Abs}[M_-[1]]^2 + \frac{5}{2}\,\text{Abs}[M_+[1]]^2 + x\,(-2\,\text{Re}[M_-[1]\,E_+[0]^*] + 2\,\text{Re}[M_+[1]\,E_+[0]^*] + 6\,\text{Re}[E_+[0]\,E_+[1]^*]) + 3\,\text{Re}[M_-[1]\,E_+[1]^*] - 3\,\text{Re}[M_+[1]\,E_+[1]^*] + \frac{3}{2}\,x^2\,(3\,\text{Abs}[E_+[1]]^2 - \text{Abs}[M_+[1]]^2 - 6\,\text{Re}[M_-[1]\,E_+[1]^*] + 6\,\text{Re}[M_+[1]\,E_+[1]^*] - 2\,\text{Re}[M_+[1]\,M_-[1]^*]) + \text{Re}[M_+[1]\,M_-[1]^*]$$

ExpandR[R_{TT}[0], 1, 2]

$$\frac{1}{2}\,(9\,\text{Abs}[E_+[1]]^2 - 3\,(\text{Abs}[M_+[1]]^2 + 2\,(-\text{Re}[M_-[1]\,E_+[1]^*] + \text{Re}[M_+[1]\,E_+[1]^*] + \text{Re}[M_+[1]\,M_-[1]^*])))\,\text{Sin}[\theta]^2$$

ExpandR[R_{LT}[0], 1, 1]

$$\left(-\frac{1}{q}\,(\sqrt{2}\,Q\,(\text{Re}[E_+[0]\,S_-[1]^*] + 3\,\text{Re}[E_+[1]\,S_+[0]^*] + \text{Re}[M_-[1]\,S_+[0]^*] - \text{Re}[M_+[1]\,S_+[0]^*] - 2\,\text{Re}[E_+[0]\,S_+[1]^*])) - \frac{1}{q}\,(6\,\sqrt{2}\,Q\,x\,(\text{Re}[E_+[1]\,S_-[1]^*] + \text{Re}[E_+[1]\,S_+[1]^*] + \text{Re}[M_-[1]\,S_+[1]^*] - \text{Re}[M_+[1]\,S_+[1]^*]))\right)\,\text{Sin}[\theta]$$

ExpandR[R_{LTh}[0], 1, 1]

$$\left(\frac{1}{q}\,(\sqrt{2}\,Q\,(\text{Im}[E_+[0]\,S_-[1]^*] + 3\,\text{Im}[E_+[1]\,S_+[0]^*] + \text{Im}[M_-[1]\,S_+[0]^*] - \text{Im}[M_+[1]\,S_+[0]^*] - 2\,\text{Im}[E_+[0]\,S_+[1]^*])) + \frac{1}{q}\,(6\,\sqrt{2}\,Q\,x\,(\text{Im}[E_+[1]\,S_-[1]^*] + \text{Im}[E_+[1]\,S_+[1]^*] + \text{Im}[M_-[1]\,S_+[1]^*] - \text{Im}[M_+[1]\,S_+[1]^*]))\right)\,\text{Sin}[\theta]$$

$mpR_{LTh}[N]$ = ExpandR[$R_{LTh}[N]$, 1]

$$\frac{1}{q}\left(\sqrt{2}\,Q\,x\,(\mathrm{Re}[E_+[0]\,S_-[1]^*] + 3\,\mathrm{Re}[E_+[1]\,S_+[0]^*] - \mathrm{Re}[M_-[1]\,S_+[0]^*] + \mathrm{Re}[M_+[1]\,S_+[0]^*] + 4\,\mathrm{Re}[E_+[0]\,S_+[1]^*])\right) + \frac{1}{q}\left(\sqrt{2}\,Q\,(-3\,\mathrm{Re}[E_+[1]\,S_-[1]^*] - \mathrm{Re}[M_-[1]\,S_-[1]^*] + \mathrm{Re}[M_+[1]\,S_-[1]^*] + \mathrm{Re}[E_+[0]\,S_+[0]^*] + 6\,\mathrm{Re}[E_+[1]\,S_+[1]^*] + 2\,\mathrm{Re}[M_-[1]\,S_+[1]^*] - 2\,\mathrm{Re}[M_+[1]\,S_+[1]^*])\right) + \frac{1}{q}\left(6\sqrt{2}\,Q\,x^2\,(\mathrm{Re}[E_+[1]\,S_-[1]^*] + \mathrm{Re}[E_+[1]\,S_+[1]^*] - \mathrm{Re}[M_-[1]\,S_+[1]^*] + \mathrm{Re}[M_+[1]\,S_+[1]^*])\right)$$

$mpR_{LTh}[L]$ = ExpandR[$R_{LTh}[L]$, 1, 1]

$$\left(-\frac{3\sqrt{2}\,Q\,x\,(\mathrm{Re}[E_+[1]\,S_+[0]^*] + \mathrm{Re}[M_+[1]\,S_+[0]^*] + 2\,\mathrm{Re}[E_+[0]\,S_+[1]^*])}{q} - \frac{1}{q}\left(\sqrt{2}\,Q\,(\mathrm{Re}[M_-[1]\,S_-[1]^*] + 2\,\mathrm{Re}[M_+[1]\,S_-[1]^*] + \mathrm{Re}[E_+[0]\,S_+[0]^*] - 2\,\mathrm{Re}[M_-[1]\,S_+[1]^*] - 4\,\mathrm{Re}[M_+[1]\,S_+[1]^*])\right) - \frac{18\sqrt{2}\,Q\,x^2\,(\mathrm{Re}[E_+[1]\,S_+[1]^*] + \mathrm{Re}[M_+[1]\,S_+[1]^*])}{q}\right)\mathrm{Sin}[\theta]$$

$mpR_{LTh}[S]$ = ExpandR[$R_{LTh}[S]$, 1]

$$\frac{3\sqrt{2}\,Q\,x^2\,(\mathrm{Re}[E_+[1]\,S_+[0]^*] + \mathrm{Re}[M_+[1]\,S_+[0]^*] + 2\,\mathrm{Re}[E_+[0]\,S_+[1]^*])}{q} + \frac{1}{q}\left(\sqrt{2}\,Q\,(\mathrm{Re}[E_+[0]\,S_-[1]^*] - \mathrm{Re}[M_-[1]\,S_+[0]^*] - 2\,(\mathrm{Re}[M_+[1]\,S_+[0]^*] + \mathrm{Re}[E_+[0]\,S_+[1]^*]))\right) + \frac{1}{q}\left(\sqrt{2}\,Q\,x\,(3\,\mathrm{Re}[E_+[1]\,S_-[1]^*] - \mathrm{Re}[M_-[1]\,S_-[1]^*] + \mathrm{Re}[M_+[1]\,S_-[1]^*] + \mathrm{Re}[E_+[0]\,S_+[0]^*] - 6\,\mathrm{Re}[E_+[1]\,S_+[1]^*] - 4\,\mathrm{Re}[M_-[1]\,S_+[1]^*] - 14\,\mathrm{Re}[M_+[1]\,S_+[1]^*])\right) + \frac{18\sqrt{2}\,Q\,x^3\,(\mathrm{Re}[E_+[1]\,S_+[1]^*] + \mathrm{Re}[M_+[1]\,S_+[1]^*])}{q}$$

$mpR_{TTh}[L]$ = ExpandR[$R_{TTh}[L]$, 1]

$$2\,\mathrm{Re}[M_-[1]\,E_+[0]^*] + \mathrm{Re}[M_+[1]\,E_+[0]^*] + 3\,\mathrm{Re}[E_+[0]\,E_+[1]^*] - 3\,x^2\,(\mathrm{Re}[M_+[1]\,E_+[0]^*] + 3\,\mathrm{Re}[E_+[0]\,E_+[1]^*]) - 18\,x^3\,(\mathrm{Abs}[E_+[1]]^2 + \mathrm{Re}[M_+[1]\,E_+[1]^*]) + x\,(-\mathrm{Abs}[E_+[0]]^2 + 9\,\mathrm{Abs}[E_+[1]]^2 - \mathrm{Abs}[M_-[1]]^2 - \mathrm{Abs}[M_+[1]]^2 + 6\,\mathrm{Re}[M_-[1]\,E_+[1]^*] + 12\,\mathrm{Re}[M_+[1]\,E_+[1]^*] + 2\,\mathrm{Re}[M_+[1]\,M_-[1]^*])$$

$\text{mpR}_{\text{TTh}}[\text{S}] = \text{ExpandR}[\text{R}_{\text{TTh}}[\text{S}], 1, 1]$ // FullSimplify

$$(-\text{Abs}[E_+[0]]^2 + \text{Abs}[M_-[1]]^2 - 2\,\text{Abs}[M_+[1]]^2 + 3\,\text{Re}[M_-[1]\,E_+[1]^*] + 6\,\text{Re}[M_+[1]\,E_+[1]^*] - 3\,x\,(\text{Re}[M_+[1]\,E_+[0]^*] + 3\,\text{Re}[E_+[0]\,E_+[1]^*] + 6\,x\,(\text{Abs}[E_+[1]]^2 + \text{Re}[M_+[1]\,E_+[1]^*])) + \text{Re}[M_+[1]\,M_-[1]^*])\,\text{Sin}[\theta]$$

$\text{ExpandR}[\text{R}_{\text{L}}[\text{N}], 1, 1]$ // FullSimplify

$$\frac{2\,Q^2\,(\text{Im}[S_-[1]^*\,S_+[0]] - 2\,\text{Im}[S_+[1]^*\,S_+[0]] + 6\,x\,\text{Im}[S_-[1]^*\,S_+[1]])\,\text{Sin}[\theta]}{q^2}$$

$\text{ExpandR}[\text{R}_{\text{T}}[\text{N}], 1, 1]$ // FullSimplify

$$(2\,\text{Im}[M_-[1]\,E_+[0]^*] + \text{Im}[M_+[1]\,E_+[0]^*] - 3\,(\text{Im}[E_+[0]\,E_+[1]^*] - 3\,x\,\text{Im}[M_-[1]\,E_+[1]^*] + x\,\text{Im}[M_+[1]\,M_-[1]^*]))\,\text{Sin}[\theta]$$

$\text{ExpandR}[\text{R}_{\text{LT}}[\text{N}], 1]$ // FullSimplify

$$\frac{1}{q}(\sqrt{2}\,Q\,(-3\,\text{Im}[E_+[1]\,S_-[1]^*] - \text{Im}[M_-[1]\,S_-[1]^*] + \text{Im}[M_+[1]\,S_-[1]^*] + \text{Im}[E_+[0]\,S_+[0]^*] + 6\,\text{Im}[E_+[1]\,S_+[1]^*] + 2\,\text{Im}[M_-[1]\,S_+[1]^*] - 2\,\text{Im}[M_+[1]\,S_+[1]^*] + x\,(\text{Im}[E_+[0]\,S_-[1]^*] + 3\,\text{Im}[E_+[1]\,S_+[0]^*] - \text{Im}[M_-[1]\,S_+[0]^*] + \text{Im}[M_+[1]\,S_+[0]^*] + 4\,\text{Im}[E_+[0]\,S_+[1]^*] + 6\,x\,(\text{Im}[E_+[1]\,S_-[1]^*] + \text{Im}[E_+[1]\,S_+[1]^*] - \text{Im}[M_-[1]\,S_+[1]^*] + \text{Im}[M_+[1]\,S_+[1]^*]))))$$

$\text{ExpandR}[\text{R}_{\text{LT}}[\text{L}], 1, 1]$ // FullSimplify

$$\frac{1}{q}(\sqrt{2}\,Q\,(\text{Im}[M_-[1]\,S_-[1]^*] + 2\,\text{Im}[M_+[1]\,S_-[1]^*] + \text{Im}[E_+[0]\,S_+[0]^*] - 2\,(\text{Im}[M_-[1]\,S_+[1]^*] + 2\,\text{Im}[M_+[1]\,S_+[1]^*]) + 3\,x\,(\text{Im}[E_+[1]\,S_+[0]^*] + \text{Im}[M_+[1]\,S_+[0]^*] + 2\,(\text{Im}[E_+[0]\,S_+[1]^*] + 3\,x\,(\text{Im}[E_+[1]\,S_+[1]^*] + \text{Im}[M_+[1]\,S_+[1]^*]))))\,\text{Sin}[\theta])$$

$\text{ExpandR}[\text{R}_{\text{LT}}[\text{S}], 1, 0]$ // FullSimplify

$$-\frac{1}{q}(\sqrt{2}\,Q\,(\text{Im}[E_+[0]\,S_-[1]^*] - \text{Im}[M_-[1]\,S_+[0]^*] - 2\,(\text{Im}[M_+[1]\,S_+[0]^*] + \text{Im}[E_+[0]\,S_+[1]^*]) + x\,(3\,\text{Im}[E_+[1]\,S_-[1]^*] - \text{Im}[M_-[1]\,S_-[1]^*] + \text{Im}[M_+[1]\,S_-[1]^*] + \text{Im}[E_+[0]\,S_+[0]^*] - 2\,(3\,\text{Im}[E_+[1]\,S_+[1]^*] + 2\,\text{Im}[M_-[1]\,S_+[1]^*] + 7\,\text{Im}[M_+[1]\,S_+[1]^*]) + 3\,x\,(\text{Im}[E_+[1]\,S_+[0]^*] + \text{Im}[M_+[1]\,S_+[0]^*] + 2\,(\text{Im}[E_+[0]\,S_+[1]^*] + 3\,x\,(\text{Im}[E_+[1]\,S_+[1]^*] + \text{Im}[M_+[1]\,S_+[1]^*]))))))$$

$\text{ExpandR}[\text{R}_{\text{TT}}[\text{N}], 1, 1]$ // FullSimplify

$$(-3\,(\text{Im}[M_+[1]\,E_+[0]^*] + \text{Im}[E_+[0]\,E_+[1]^*]) + 3\,x\,(\text{Im}[M_-[1]\,E_+[1]^*] - 4\,\text{Im}[M_+[1]\,E_+[1]^*] + \text{Im}[M_+[1]\,M_-[1]^*]))\,\text{Sin}[\theta]$$

```
ExpandR[R_TT[L], 1, 2] // FullSimplify
```

$$-3\,(\text{Im}[M_+[1]\,E_+[0]^*] + \text{Im}[E_+[0]\,E_+[1]^*] + 6\,x\,\text{Im}[M_+[1]\,E_+[1]^*])\,\text{Sin}[\theta]^2$$

```
ExpandR[R_TT[S], 1, 1] // FullSimplify
```

$$(3\,x\,(\text{Im}[M_+[1]\,E_+[0]^*] + \text{Im}[E_+[0]\,E_+[1]^*] + 6\,x\,\text{Im}[M_+[1]\,E_+[1]^*]) - 3\,(\text{Im}[M_-[1]\,E_+[1]^*] + 2\,\text{Im}[M_+[1]\,E_+[1]^*] + \text{Im}[M_+[1]\,M_-[1]^*]))\,\text{Sin}[\theta]$$

■ P_{33} dominance

Here we display expansions based upon Δ dominance, retaining only *s* and *p* waves.

```
mpBrief[P_33, R_, n_Integer: 0] :=
 AbbreviateMultipoleExpansion[ExpandR[R, 1, n], {M_+[1], E_+[1], S_+[1]}]
```

```
mpBrief[P_33, R_L[0]]
```

$$\frac{8\,Q^2\,x\,\text{Re}[S_+[1]^*\,S_+[0]]}{q^2} + \frac{4\,Q^2\,(\text{Abs}[S_+[1]]^2 - \text{Re}[S_-[1]^*\,S_+[1]])}{q^2} + \frac{12\,Q^2\,x^2\,(\text{Abs}[S_+[1]]^2 + \text{Re}[S_-[1]^*\,S_+[1]])}{q^2}$$

```
mpBrief[P_33, R_T[0]]
```

$$\frac{9}{2}\,\text{Abs}[E_+[1]]^2 + \frac{5}{2}\,\text{Abs}[M_+[1]]^2 + 2\,x\,(\text{Re}[M_+[1]\,E_+[0]^*] + 3\,\text{Re}[E_+[0]\,E_+[1]^*]) + 3\,\text{Re}[M_-[1]\,E_+[1]^*] - 3\,\text{Re}[M_+[1]\,E_+[1]^*] + \frac{3}{2}\,x^2\,(3\,\text{Abs}[E_+[1]]^2 - \text{Abs}[M_+[1]]^2 - 6\,\text{Re}[M_-[1]\,E_+[1]^*] + 6\,\text{Re}[M_+[1]\,E_+[1]^*] - 2\,\text{Re}[M_+[1]\,M_-[1]^*]) + \text{Re}[M_+[1]\,M_-[1]^*]$$

```
mpBrief[P_33, R_LT[0], 1]
```

$$-\frac{\sqrt{2}\,Q\,(3\,\text{Re}[E_+[1]\,S_+[0]^*] - \text{Re}[M_+[1]\,S_+[0]^*] - 2\,\text{Re}[E_+[0]\,S_+[1]^*])\,\text{Sin}[\theta]}{q} - \frac{1}{q}(6\sqrt{2}\,Q\,x\,(\text{Re}[E_+[1]\,S_-[1]^*] + \text{Re}[E_+[1]\,S_+[1]^*] + \text{Re}[M_-[1]\,S_+[1]^*] - \text{Re}[M_+[1]\,S_+[1]^*])\,\text{Sin}[\theta])$$

```
mpBrief[P_33, R_TT[0], 2]
```

$$\frac{3}{2}\,(3\,\text{Abs}[E_+[1]]^2 - \text{Abs}[M_+[1]]^2 - 2\,(-\text{Re}[M_-[1]\,E_+[1]^*] + \text{Re}[M_+[1]\,E_+[1]^*] + \text{Re}[M_+[1]\,M_-[1]^*]))\,\text{Sin}[\theta]^2$$

mpBrief[P_{33}, R_{TT}[S], 1] // Simplify

$3\,(x\,(\mathrm{Im}[M_+[1]\,E_+[0]^*] + \mathrm{Im}[E_+[0]\,E_+[1]^*]) - \mathrm{Im}[M_-[1]\,E_+[1]^*] - 2\,\mathrm{Im}[M_+[1]\,E_+[1]^*] + 6\,x^2\,\mathrm{Im}[M_+[1]\,E_+[1]^*] - \mathrm{Im}[M_+[1]\,M_-[1]^*])\,\mathrm{Sin}[\theta]$

mpBrief[P_{33}, R_{LT}[N]]

$$\frac{\sqrt{2}\,Q\,x\,(3\,\mathrm{Im}[E_+[1]\,S_+[0]^*] + \mathrm{Im}[M_+[1]\,S_+[0]^*] + 4\,\mathrm{Im}[E_+[0]\,S_+[1]^*])}{q} + \frac{1}{q}(\sqrt{2}\,Q\,(-3\,\mathrm{Im}[E_+[1]\,S_-[1]^*] + \mathrm{Im}[M_+[1]\,S_-[1]^*] + 6\,\mathrm{Im}[E_+[1]\,S_+[1]^*] + 2\,\mathrm{Im}[M_-[1]\,S_+[1]^*] - 2\,\mathrm{Im}[M_+[1]\,S_+[1]^*])) + \frac{1}{q}(6\,\sqrt{2}\,Q\,x^2\,(\mathrm{Im}[E_+[1]\,S_-[1]^*] + \mathrm{Im}[E_+[1]\,S_+[1]^*] - \mathrm{Im}[M_-[1]\,S_+[1]^*] + \mathrm{Im}[M_+[1]\,S_+[1]^*]))$$

mpBrief[P_{33}, R_{LT}[S]]

$$\frac{2\,\sqrt{2}\,Q\,(\mathrm{Im}[M_+[1]\,S_+[0]^*] + \mathrm{Im}[E_+[0]\,S_+[1]^*])}{q} - \frac{3\,\sqrt{2}\,Q\,x^2\,(\mathrm{Im}[E_+[1]\,S_+[0]^*] + \mathrm{Im}[M_+[1]\,S_+[0]^*] + 2\,\mathrm{Im}[E_+[0]\,S_+[1]^*])}{q} - \frac{18\,\sqrt{2}\,Q\,x^3\,(\mathrm{Im}[E_+[1]\,S_+[1]^*] + \mathrm{Im}[M_+[1]\,S_+[1]^*])}{q} - \frac{1}{q}(\sqrt{2}\,Q\,x\,(3\,\mathrm{Im}[E_+[1]\,S_-[1]^*] + \mathrm{Im}[M_+[1]\,S_-[1]^*] - 2\,(3\,\mathrm{Im}[E_+[1]\,S_+[1]^*] + 2\,\mathrm{Im}[M_-[1]\,S_+[1]^*] + 7\,\mathrm{Im}[M_+[1]\,S_+[1]^*])))$$

mpBrief[P_{33}, R_{LTh}[S]]

$$-\frac{2\,\sqrt{2}\,Q\,(\mathrm{Re}[M_+[1]\,S_+[0]^*] + \mathrm{Re}[E_+[0]\,S_+[1]^*])}{q} + \frac{3\,\sqrt{2}\,Q\,x^2\,(\mathrm{Re}[E_+[1]\,S_+[0]^*] + \mathrm{Re}[M_+[1]\,S_+[0]^*] + 2\,\mathrm{Re}[E_+[0]\,S_+[1]^*])}{q} + \frac{18\,\sqrt{2}\,Q\,x^3\,(\mathrm{Re}[E_+[1]\,S_+[1]^*] + \mathrm{Re}[M_+[1]\,S_+[1]^*])}{q} + \frac{1}{q}(\sqrt{2}\,Q\,x\,(3\,\mathrm{Re}[E_+[1]\,S_-[1]^*] + \mathrm{Re}[M_+[1]\,S_-[1]^*] - 2\,(3\,\mathrm{Re}[E_+[1]\,S_+[1]^*] + 2\,\mathrm{Re}[M_-[1]\,S_+[1]^*] + 7\,\mathrm{Re}[M_+[1]\,S_+[1]^*])))$$

mpBrief[P_{33}, R_{LTh}[L], 1] // Simplify

$$-\frac{1}{q}(\sqrt{2}\,Q\,(2\,\mathrm{Re}[M_+[1]\,S_-[1]^*] + 3\,x\,\mathrm{Re}[E_+[1]\,S_+[0]^*] + 3\,x\,\mathrm{Re}[M_+[1]\,S_+[0]^*] + 6\,x\,\mathrm{Re}[E_+[0]\,S_+[1]^*] + 18\,x^2\,\mathrm{Re}[E_+[1]\,S_+[1]^*] - 2\,\mathrm{Re}[M_-[1]\,S_+[1]^*] - 4\,\mathrm{Re}[M_+[1]\,S_+[1]^*] + 18\,x^2\,\mathrm{Re}[M_+[1]\,S_+[1]^*])\,\mathrm{Sin}[\theta])$$

```
mpBrief[P33, RTTh[S], 1] // Simplify
```

```
(-2 Abs[M+[1]]^2 - 3 x (Re[M+[1] E+[0]*] + 3 Re[E+[0] E+[1]*]) +
   3 Re[M-[1] E+[1]*] + 6 Re[M+[1] E+[1]*] -
   18 x^2 (Abs[E+[1]]^2 + Re[M+[1] E+[1]*]) + Re[M+[1] M-[1]*]) Sin[θ]
```

```
mpBrief[P33, RTTh[L]]
```

```
Re[M+[1] E+[0]*] + 3 Re[E+[0] E+[1]*] -
 3 x^2 (Re[M+[1] E+[0]*] + 3 Re[E+[0] E+[1]*]) -
 18 x^3 (Abs[E+[1]]^2 + Re[M+[1] E+[1]*]) + x (9 Abs[E+[1]]^2 - Abs[M+[1]]^2 +
    2 (3 Re[M-[1] E+[1]*] + 6 Re[M+[1] E+[1]*] + Re[M+[1] M-[1]*]))
```

It is also useful to examine severely truncated multipole expansions for parallel versus antiparallel kinematics. Here we assume M_{1+} dominance and define a few functions which facilitate comparisons between parallel ($\theta_\pi = \pi$) and antiparallel ($\theta_\pi = 0$) kinematics.

```
mpVeryBrief[P33, R_, n_Integer: 0] :=
 AbbreviateMultipoleExpansion[ExpandR[R, 1, n], {M+[1]}]

fbsum[P33, R_, n_Integer: 0] :=
        (mpVeryBrief[P33, R, n] + mpVeryBrief[P33, R, n] /. x → -x) //
  Simplify

fbdiff[P33, R_, n_Integer: 0] :=
        (mpVeryBrief[P33, R, n] - mpVeryBrief[P33, R, n] /. x → -x) //
  Simplify

fb[P33, R_, n_Integer: 0] := Module[{f, b},
        f = mpVeryBrief[P33, R, n] /. x → -x;
        b = mpVeryBrief[P33, R, n];
        (f - b) / (f + b) // Simplify]
```

Selected response functions for M_{1+} dominance are given below.

```
mpVeryBrief[P33, RT[0]]
```

$$\frac{5}{2}\,\mathrm{Abs}[M_+[1]]^2 + 2\,x\,\mathrm{Re}[M_+[1]\,E_+[0]^*] - 3\,\mathrm{Re}[M_+[1]\,E_+[1]^*] + \mathrm{Re}[M_+[1]\,M_-[1]^*] - \frac{3}{2}\,x^2\,(\mathrm{Abs}[M_+[1]]^2 - 6\,\mathrm{Re}[M_+[1]\,E_+[1]^*] + 2\,\mathrm{Re}[M_+[1]\,M_-[1]^*])$$

```
mpVeryBrief[P33, RLT[0], 1] // FullSimplify
```

$$\frac{\sqrt{2}\,Q\,(\mathrm{Re}[M_+[1]\,S_+[0]^*] + 6\,x\,\mathrm{Re}[M_+[1]\,S_+[1]^*])\,\mathrm{Sin}[\theta]}{q}$$

```
mpVeryBrief[P33, RTT[0], 2] // FullSimplify
```

$$-\frac{3}{2}\left(\text{Abs}[M_+[1]]^2+2\left(\text{Re}[M_+[1]\,E_+[1]^*]+\text{Re}[M_+[1]\,M_-[1]^*]\right)\right)\text{Sin}[\theta]^2$$

```
mpVeryBrief[P33, RLT[N]] // FullSimplify
```

$$\frac{\sqrt{2}\,Q\left(\text{Im}[M_+[1]\,S_-[1]^*]+x\,\text{Im}[M_+[1]\,S_+[0]^*]+2\,(-1+3\,x^2)\,\text{Im}[M_+[1]\,S_+[1]^*]\right)}{q}$$

```
mpVeryBrief[P33, RLTh[S]] // FullSimplify
```

$$\frac{1}{q}\left(\sqrt{2}\,Q\left(x\,\text{Re}[M_+[1]\,S_-[1]^*]+(-2+3\,x^2)\,\text{Re}[M_+[1]\,S_+[0]^*]+2\,x\,(-7+9\,x^2)\,\text{Re}[M_+[1]\,S_+[1]^*]\right)\right)$$

```
mpVeryBrief[P33, RTTh[S], 1] // FullSimplify
```

$$\left(-2\,\text{Abs}[M_+[1]]^2-3\,x\,\text{Re}[M_+[1]\,E_+[0]^*]+(6-18\,x^2)\,\text{Re}[M_+[1]\,E_+[1]^*]+\text{Re}[M_+[1]\,M_-[1]^*]\right)\text{Sin}[\theta]$$

```
mpVeryBrief[P33, RLTh[L], 1] // FullSimplify
```

$$-\frac{1}{q}\left(\sqrt{2}\,Q\left(2\,\text{Re}[M_+[1]\,S_-[1]^*]+3\,x\,\text{Re}[M_+[1]\,S_+[0]^*]+2\,(-2+9\,x^2)\,\text{Re}[M_+[1]\,S_+[1]^*]\right)\text{Sin}[\theta]\right)$$

```
mpVeryBrief[P33, RTTh[L]] // FullSimplify
```

$$-x\,\text{Abs}[M_+[1]]^2+(1-3\,x^2)\,\text{Re}[M_+[1]\,E_+[0]^*]+2\,x\left((6-9\,x^2)\,\text{Re}[M_+[1]\,E_+[1]^*]+\text{Re}[M_+[1]\,M_-[1]^*]\right)$$

Observing that the S_{0+} and S_{1+} contributions to $R_{\rm LT}$ have opposite symmetries with respect to $\theta\to\pi-\theta$, we can separate those terms using

```
fbsum[P33, RLT[0], 1] // FullSimplify
```

$$\frac{2\sqrt{2}\,Q\left(\text{Re}[M_+[1]\,S_+[0]^*]-6\,x\,\text{Re}[M_+[1]\,S_+[1]^*]\right)\text{Sin}[\theta]}{q}$$

```
fbdiff[P33, RLT[0]]
```

0

Similarly, observing that the S_{0+} contribution to $R_{\rm LTh}[S]$ changes sign relative to the S_{1-} and S_{1+} terms

```
mpVeryBrief[P33, RLTh[S]] /. x → -x
```

$$-\frac{2\sqrt{2}\,Q\,\text{Re}[M_+[1]\,S_+[0]^*]}{q}+\frac{3\sqrt{2}\,Q\,x^2\,\text{Re}[M_+[1]\,S_+[0]^*]}{q}-\frac{\sqrt{2}\,Q\,x\,(\text{Re}[M_+[1]\,S_-[1]^*]-14\,\text{Re}[M_+[1]\,S_+[1]^*])}{q}-\frac{18\sqrt{2}\,Q\,x^3\,\text{Re}[M_+[1]\,S_+[1]^*]}{q}$$

```
mpVeryBrief[P33, RLTh[S]]
```

$$-\frac{2\sqrt{2}\,Q\,\text{Re}[M_+[1]\,S_+[0]^*]}{q}+\frac{3\sqrt{2}\,Q\,x^2\,\text{Re}[M_+[1]\,S_+[0]^*]}{q}+\frac{\sqrt{2}\,Q\,x\,(\text{Re}[M_+[1]\,S_-[1]^*]-14\,\text{Re}[M_+[1]\,S_+[1]^*])}{q}+\frac{18\sqrt{2}\,Q\,x^3\,\text{Re}[M_+[1]\,S_+[1]^*]}{q}$$

we find that the S_{0+} contribution can be extracted by averaging $R_{\text{LTh}}[S]$ for parallel and antiparallel kinematics.

```
fbsum[P33, RLTh[S]] // FullSimplify
```

$$-\frac{1}{q}(2\sqrt{2}\,Q\,(x\,\text{Re}[M_+[1]\,S_-[1]^*]+(2-3\,x^2)\,\text{Re}[M_+[1]\,S_+[0]^*]+2\,x\,(-7+9\,x^2)\,\text{Re}[M_+[1]\,S_+[1]^*]))$$

```
fbdiff[P33, RLTh[S]]
```

0

Finally,

```
mpVeryBrief[P33, RTTh[L]]
```

$$\text{Re}[M_+[1]\,E_+[0]^*]-3\,x^2\,\text{Re}[M_+[1]\,E_+[0]^*]-18\,x^3\,\text{Re}[M_+[1]\,E_+[1]^*]+x\,(-\text{Abs}[M_+[1]]^2+2\,(6\,\text{Re}[M_+[1]\,E_+[1]^*]+\text{Re}[M_+[1]\,M_-[1]^*]))$$

```
fbsum[P33, RTTh[L]]
```

$$2\,(x\,\text{Abs}[M_+[1]]^2+(1-3\,x^2)\,\text{Re}[M_+[1]\,E_+[0]^*]+2\,x\,((-6+9\,x^2)\,\text{Re}[M_+[1]\,E_+[1]^*]-\text{Re}[M_+[1]\,M_-[1]^*]))$$

```
fbdiff[P33, RTTh[L]]
```

0

the E_{0+} contribution can be obtained by averaging $R_{\text{TTh}}[L]$. Alternatively, the following expressions

```
fbsum[P33, RLTh[L], 1]
```

$$\frac{1}{q}\left(2\sqrt{2}\,Q\left(-2\,\mathrm{Re}[M_+[1]\,S_-[1]^*]+3\,x\,\mathrm{Re}[M_+[1]\,S_+[0]^*]+2\,(2-9\,x^2)\,\mathrm{Re}[M_+[1]\,S_+[1]^*]\right)\mathrm{Sin}[\theta]\right)$$

```
fbsum[P33, RTTh[S], 1]
```

$$2\left(-2\,\mathrm{Abs}[M_+[1]]^2+3\,x\,\mathrm{Re}[M_+[1]\,E_+[0]^*]+6\,\mathrm{Re}[M_+[1]\,E_+[1]^*]-18\,x^2\,\mathrm{Re}[M_+[1]\,E_+[1]^*]+\mathrm{Re}[M_+[1]\,M_-[1]^*]\right)\mathrm{Sin}[\theta]$$

yield redundant determinations of these interference products; this redundancy can be used to test for and minimize model dependence.

However, we must still assess the contributions of other multipoles more quantitatively.

```
1/2 (mpBrief[P33, RLTh[S], 0] + mpBrief[P33, RLTh[S]] /. x → -x) // Simplify
```

$$\frac{1}{q}\Big(\sqrt{2}\,Q\,\big(-3\,x\,\mathrm{Re}[E_+[1]\,S_-[1]^*]-x\,\mathrm{Re}[M_+[1]\,S_-[1]^*]+3\,x^2\,\mathrm{Re}[E_+[1]\,S_+[0]^*]-2\,\mathrm{Re}[M_+[1]\,S_+[0]^*]+3\,x^2\,\mathrm{Re}[M_+[1]\,S_+[0]^*]-2\,\mathrm{Re}[E_+[0]\,S_+[1]^*]+6\,x^2\,\mathrm{Re}[E_+[0]\,S_+[1]^*]+6\,x\,\mathrm{Re}[E_+[1]\,S_+[1]^*]-18\,x^3\,\mathrm{Re}[E_+[1]\,S_+[1]^*]+4\,x\,\mathrm{Re}[M_-[1]\,S_+[1]^*]+14\,x\,\mathrm{Re}[M_+[1]\,S_+[1]^*]-18\,x^3\,\mathrm{Re}[M_+[1]\,S_+[1]^*]\big)\Big)$$

```
1/2 (mpBrief[P33, RTTh[L], 0] + mpBrief[P33, RTTh[L]] /. x → -x) // Simplify
```

$$\frac{1}{2}\Big(2\,\mathrm{Re}[M_+[1]\,E_+[0]^*]+6\,\mathrm{Re}[E_+[0]\,E_+[1]^*]-6\,x^2\,(\mathrm{Re}[M_+[1]\,E_+[0]^*]+3\,\mathrm{Re}[E_+[0]\,E_+[1]^*])+36\,x^3\,(\mathrm{Abs}[E_+[1]]^2+\mathrm{Re}[M_+[1]\,E_+[1]^*])-2\,x\,\big(9\,\mathrm{Abs}[E_+[1]]^2-\mathrm{Abs}[M_+[1]]^2+2\,(3\,\mathrm{Re}[M_-[1]\,E_+[1]^*]+6\,\mathrm{Re}[M_+[1]\,E_+[1]^*]+\mathrm{Re}[M_+[1]\,M_-[1]^*])\big)\Big)$$

More generally, it is advantageous to measure response functions at complementary angles in order to exploit the fact that even and odd partial waves have opposite symmetries with respect to $\theta \to \pi - \theta$.

■ S_{11} dominance

Here I compare multipole expansions for S_{11} dominance with results from KDT, retaining just S_{11}, P_{11}, and D_{13} multipoles. Note that my $\{\hat{S}, \hat{N}, \hat{L}\}$ correspond to their $\{-\hat{x}', \hat{y}', -\hat{z}'\}$.

```
mpBrief[S11, R_, n_Integer: 0] :=
 AbbreviateMultipoleExpansion[ExpandR[R, 2, n], {E+[0], S+[0]}]
```

mpBrief[S_{11}, R_L[0]] // FullSimplify

$$\frac{1}{q^2}\left(Q^2\left(\text{Abs}[S_+[0]]^2+2\,x\,\text{Re}[S_-[1]^*\,S_+[0]]+4\,(-1+3\,x^2)\,\text{Re}[S_-[2]^*\,S_+[0]]+8\,x\,\text{Re}[S_+[1]^*\,S_+[0]]+9\,(-1+3\,x^2)\,\text{Re}[S_+[2]^*\,S_+[0]]\right)\right)$$

mpBrief[S_{11}, R_T[0]]

$$\text{Abs}[E_+[0]]^2-\text{Re}[E_+[0]\,E_-[2]^*]+3\,\text{Re}[M_-[2]\,E_+[0]^*]-3\,\text{Re}[M_+[2]\,E_+[0]^*]+x\,(-2\,\text{Re}[M_-[1]\,E_+[0]^*]+2\,\text{Re}[M_+[1]\,E_+[0]^*]+6\,\text{Re}[E_+[0]\,E_+[1]^*])-6\,\text{Re}[E_+[0]\,E_+[2]^*]+3\,x^2\,(\text{Re}[E_+[0]\,E_-[2]^*]-3\,\text{Re}[M_-[2]\,E_+[0]^*]+3\,\text{Re}[M_+[2]\,E_+[0]^*]+6\,\text{Re}[E_+[0]\,E_+[2]^*])$$

mpBrief[S_{11}, R_{LT}[0], 1] // Simplify

$$-\frac{1}{q}\left(\sqrt{2}\,Q\,(\text{Re}[E_+[0]\,S_-[1]^*]+6\,x\,\text{Re}[E_+[0]\,S_-[2]^*]-3\,x\,\text{Re}[E_-[2]\,S_+[0]^*]+3\,\text{Re}[E_+[1]\,S_+[0]^*]+12\,x\,\text{Re}[E_+[2]\,S_+[0]^*]+\text{Re}[M_-[1]\,S_+[0]^*]+3\,x\,\text{Re}[M_-[2]\,S_+[0]^*]-\text{Re}[M_+[1]\,S_+[0]^*]-3\,x\,\text{Re}[M_+[2]\,S_+[0]^*]-2\,\text{Re}[E_+[0]\,S_+[1]^*]-9\,x\,\text{Re}[E_+[0]\,S_+[2]^*])\,\text{Sin}[\theta]\right)$$

mpBrief[S_{11}, R_{TT}[0], 2]

$$-3\,(\text{Re}[E_+[0]\,E_-[2]^*]+\text{Re}[M_-[2]\,E_+[0]^*]-\text{Re}[M_+[2]\,E_+[0]^*]+\text{Re}[E_+[0]\,E_+[2]^*])\,\text{Sin}[\theta]^2$$

mpBrief[S_{11}, R_{LT}[N]]

$$\frac{1}{q}\left(\sqrt{2}\,Q\,x\,(\text{Im}[E_+[0]\,S_-[1]^*]+3\,\text{Im}[E_+[1]\,S_+[0]^*]-\text{Im}[M_-[1]\,S_+[0]^*]+\text{Im}[M_+[1]\,S_+[0]^*]+4\,\text{Im}[E_+[0]\,S_+[1]^*])\right)-\frac{1}{\sqrt{2}\,q}\left(Q\,(4\,\text{Im}[E_+[0]\,S_-[2]^*]+4\,\text{Im}[E_-[2]\,S_+[0]^*]-2\,\text{Im}[E_+[0]\,S_+[0]^*]+9\,\text{Im}[E_+[2]\,S_+[0]^*]+9\,\text{Im}[E_+[0]\,S_+[2]^*])\right)+\frac{1}{\sqrt{2}\,q}\left(3\,Q\,x^2\,(4\,\text{Im}[E_+[0]\,S_-[2]^*]+2\,\text{Im}[E_-[2]\,S_+[0]^*]+7\,\text{Im}[E_+[2]\,S_+[0]^*]-2\,\text{Im}[M_-[2]\,S_+[0]^*]+2\,\text{Im}[M_+[2]\,S_+[0]^*]+9\,\text{Im}[E_+[0]\,S_+[2]^*])\right)$$

mpBrief[S_{11}, R_{LTh}[S]]

$$\begin{aligned}&\frac{3\sqrt{2}\,Q\,x^2\,(\mathrm{Re}[E_+[1]\,S_+[0]^*]+\mathrm{Re}[M_+[1]\,S_+[0]^*]+2\,\mathrm{Re}[E_+[0]\,S_+[1]^*])}{q}+\\&\frac{1}{q}(\sqrt{2}\,Q\,(\mathrm{Re}[E_+[0]\,S_-[1]^*]-\mathrm{Re}[M_-[1]\,S_+[0]^*]-\\&\quad 2\,(\mathrm{Re}[M_+[1]\,S_+[0]^*]+\mathrm{Re}[E_+[0]\,S_+[1]^*])))+\frac{1}{\sqrt{2}\,q}\\&(Q\,x\,(8\,\mathrm{Re}[E_+[0]\,S_-[2]^*]+2\,\mathrm{Re}[E_-[2]\,S_+[0]^*]+2\,\mathrm{Re}[E_+[0]\,S_+[0]^*]-\\&\quad 3\,\mathrm{Re}[E_+[2]\,S_+[0]^*]-6\,\mathrm{Re}[M_-[2]\,S_+[0]^*]-\\&\quad 24\,\mathrm{Re}[M_+[2]\,S_+[0]^*]-27\,\mathrm{Re}[E_+[0]\,S_+[2]^*]))+\\&\frac{15\,Q\,x^3\,(\mathrm{Re}[E_+[2]\,S_+[0]^*]+2\,\mathrm{Re}[M_+[2]\,S_+[0]^*]+3\,\mathrm{Re}[E_+[0]\,S_+[2]^*])}{\sqrt{2}\,q}\end{aligned}$$

mpBrief[S_{11}, R_{LTh}[L], 1] // Simplify

$$\begin{aligned}&\frac{1}{\sqrt{2}\,q}(Q\,(4\,\mathrm{Re}[E_+[0]\,S_-[2]^*]-2\,\mathrm{Re}[E_-[2]\,S_+[0]^*]-\\&\quad 2\,\mathrm{Re}[E_+[0]\,S_+[0]^*]-6\,x\,\mathrm{Re}[E_+[1]\,S_+[0]^*]+3\,\mathrm{Re}[E_+[2]\,S_+[0]^*]-\\&\quad 15\,x^2\,\mathrm{Re}[E_+[2]\,S_+[0]^*]-6\,\mathrm{Re}[M_-[2]\,S_+[0]^*]-6\,x\,\mathrm{Re}[M_+[1]\,S_+[0]^*]+\\&\quad 6\,\mathrm{Re}[M_+[2]\,S_+[0]^*]-30\,x^2\,\mathrm{Re}[M_+[2]\,S_+[0]^*]-12\,x\,\mathrm{Re}[E_+[0]\,S_+[1]^*]+\\&\quad 9\,\mathrm{Re}[E_+[0]\,S_+[2]^*]-45\,x^2\,\mathrm{Re}[E_+[0]\,S_+[2]^*])\,\mathrm{Sin}[\theta])\end{aligned}$$

mpBrief[S_{11}, R_{TTh}[S], 1] // Simplify

$$\begin{aligned}&(-\mathrm{Abs}[E_+[0]]^2+\mathrm{Re}[E_+[0]\,E_-[2]^*]-3\,\mathrm{Re}[M_-[2]\,E_+[0]^*]+3\,\mathrm{Re}[M_+[2]\,E_+[0]^*]-\\&\quad 3\,x\,(\mathrm{Re}[M_+[1]\,E_+[0]^*]+3\,\mathrm{Re}[E_+[0]\,E_+[1]^*])+6\,\mathrm{Re}[E_+[0]\,E_+[2]^*]-\\&\quad 15\,x^2\,(\mathrm{Re}[M_+[2]\,E_+[0]^*]+2\,\mathrm{Re}[E_+[0]\,E_+[2]^*]))\,\mathrm{Sin}[\theta]\end{aligned}$$

mpBrief[S_{11}, R_{TTh}[L]]

$$\begin{aligned}&2\,\mathrm{Re}[M_-[1]\,E_+[0]^*]+\mathrm{Re}[M_+[1]\,E_+[0]^*]+\\&\ 3\,\mathrm{Re}[E_+[0]\,E_+[1]^*]-3\,x^2\,(\mathrm{Re}[M_+[1]\,E_+[0]^*]+3\,\mathrm{Re}[E_+[0]\,E_+[1]^*])-\\&\ 15\,x^3\,(\mathrm{Re}[M_+[2]\,E_+[0]^*]+2\,\mathrm{Re}[E_+[0]\,E_+[2]^*])+\\&\ x\,(-\mathrm{Abs}[E_+[0]]^2-2\,\mathrm{Re}[E_+[0]\,E_-[2]^*]+\\&\quad 6\,\mathrm{Re}[M_-[2]\,E_+[0]^*]+9\,\mathrm{Re}[M_+[2]\,E_+[0]^*]+18\,\mathrm{Re}[E_+[0]\,E_+[2]^*])\end{aligned}$$

Legendre expansions of response functions

■ expansion functions

```
ToLegendreP[x_, m_?EvenQ] :=
  Sum[(2^(2 n) (4 n + 1) m! (m/2 + n)!)/((m + 2 n + 1)! (m/2 - n)!) P_(2 n)[x], {n, 0, m/2}];
ToLegendreP[x_, m_?OddQ] :=
  Sum[(2^(2 n+1) (4 n + 3) m! ((m+1)/2 + n)!)/((m + 2 n + 2)! ((m-1)/2 - n)!) P_(2 n+1)[x], {n, 0, m/2 + 1}];
```

```
xExpand := x^n_. :→ ToLegendreP[x, n];
```

```
RtoLegendre[sp, R_, n_] :=
 Collect[MySimplify[ExpandR[R, 1, n] /. xExpand /. P0[x] → 1],
  {Sin[θ], Pλ_[x]}, MySimplify]
```

```
RtoLegendre[P33, R_, n_] :=
 Collect[MySimplify[AbbreviateMultipoleExpansion[
      ExpandR[R, 1, n], {M+[1], E+[1], S+[1]}] /. xExpand /.
    P0[x] → 1], {Sin[θ], Pλ_[x]}, MySimplify]
```

```
RtoLegendre[M1, R_, n_] := Collect[
  AbbreviateMultipoleExpansion[ExpandR[R, 1, n], {M+[1]}] /. xExpand /.
   P0[x] → 1, {Sin[θ], Pλ_[x]}, MySimplify]
```

```
RtoLegendre[R_, lmax_Integer, n_Integer] :=
 Collect[MySimplify[ExpandR[R, lmax, n] /. xExpand /. P0[x] → 1],
  {Sin[θ], Pλ_[x]}, MySimplify]
```

■ *s*, *p* expansion

```
RtoLegendre[sp, RL[0], 0]
```

$$\frac{Q^2\,(\mathrm{Abs}[S_-[1]]^2+\mathrm{Abs}[S_+[0]]^2+8\,\mathrm{Abs}[S_+[1]]^2)}{q^2}+\frac{2\,Q^2\,(\mathrm{Re}[S_-[1]^*\,S_+[0]]+4\,\mathrm{Re}[S_+[1]^*\,S_+[0]])\,P_1[x]}{q^2}+\frac{8\,Q^2\,(\mathrm{Abs}[S_+[1]]^2+\mathrm{Re}[S_-[1]^*\,S_+[1]])\,P_2[x]}{q^2}$$

RtoLegendre[sp, R_T[0], 0]

$$\text{Abs}[E_+[0]]^2 + 6\,\text{Abs}[E_+[1]]^2 + \text{Abs}[M_-[1]]^2 + 2\,\text{Abs}[M_+[1]]^2 + (-2\,\text{Re}[M_-[1]\,E_+[0]^*] + 2\,\text{Re}[M_+[1]\,E_+[0]^*] + 6\,\text{Re}[E_+[0]\,E_+[1]^*])\,P_1[x] + (3\,\text{Abs}[E_+[1]]^2 - \text{Abs}[M_+[1]]^2 - 6\,\text{Re}[M_-[1]\,E_+[1]^*] + 6\,\text{Re}[M_+[1]\,E_+[1]^*] - 2\,\text{Re}[M_+[1]\,M_-[1]^*])\,P_2[x]$$

RtoLegendre[sp, R_{TT}[0], 2] // Collect[#, {Sin[θ], $P_{\lambda\_}$[x]}] &

$$\frac{1}{2}\,(9\,\text{Abs}[E_+[1]]^2 - 3\,(\text{Abs}[M_+[1]]^2 + 2\,(-\text{Re}[M_-[1]\,E_+[1]^*] + \text{Re}[M_+[1]\,E_+[1]^*] + \text{Re}[M_+[1]\,M_-[1]^*])))\,\text{Sin}[\theta]^2$$

RtoLegendre[sp, R_{LT}[0], 1]

$$\text{Sin}[\theta]\left(-\frac{1}{q}(\sqrt{2}\,Q\,(\text{Re}[E_+[0]\,S_-[1]^*] + 3\,\text{Re}[E_+[1]\,S_+[0]^*] + \text{Re}[M_-[1]\,S_+[0]^*] - \text{Re}[M_+[1]\,S_+[0]^*] - 2\,\text{Re}[E_+[0]\,S_+[1]^*])) - \frac{1}{q}(6\sqrt{2}\,Q\,(\text{Re}[E_+[1]\,S_-[1]^*] + \text{Re}[E_+[1]\,S_+[1]^*] + \text{Re}[M_-[1]\,S_+[1]^*] - \text{Re}[M_+[1]\,S_+[1]^*])\,P_1[x])\right)$$

RtoLegendre[sp, R_{LTh}[0], 1]

$$\text{Sin}[\theta]\left(\frac{1}{q}(\sqrt{2}\,Q\,(\text{Im}[E_+[0]\,S_-[1]^*] + 3\,\text{Im}[E_+[1]\,S_+[0]^*] + \text{Im}[M_-[1]\,S_+[0]^*] - \text{Im}[M_+[1]\,S_+[0]^*] - 2\,\text{Im}[E_+[0]\,S_+[1]^*])) + \frac{1}{q}(6\sqrt{2}\,Q\,(\text{Im}[E_+[1]\,S_-[1]^*] + \text{Im}[E_+[1]\,S_+[1]^*] + \text{Im}[M_-[1]\,S_+[1]^*] - \text{Im}[M_+[1]\,S_+[1]^*])\,P_1[x])\right)$$

RtoLegendre[sp, R_{LTh}[N], 0]

$$\frac{1}{q}(\sqrt{2}\,Q\,(-\text{Re}[E_+[1]\,S_-[1]^*] - \text{Re}[M_-[1]\,S_-[1]^*] + \text{Re}[M_+[1]\,S_-[1]^*] + \text{Re}[E_+[0]\,S_+[0]^*] + 8\,\text{Re}[E_+[1]\,S_+[1]^*])) + \frac{1}{q}(\sqrt{2}\,Q\,(\text{Re}[E_+[0]\,S_-[1]^*] + 3\,\text{Re}[E_+[1]\,S_+[0]^*] - \text{Re}[M_-[1]\,S_+[0]^*] + \text{Re}[M_+[1]\,S_+[0]^*] + 4\,\text{Re}[E_+[0]\,S_+[1]^*])\,P_1[x]) + \frac{1}{q}(4\sqrt{2}\,Q\,(\text{Re}[E_+[1]\,S_-[1]^*] + \text{Re}[E_+[1]\,S_+[1]^*] - \text{Re}[M_-[1]\,S_+[1]^*] + \text{Re}[M_+[1]\,S_+[1]^*])\,P_2[x])$$

RtoLegendre[sp, R_{LTh}[L], 1]

$$\text{Sin}[\theta]\left(-\frac{1}{q}(\sqrt{2}\,Q\,(\text{Re}[M_-[1]\,S_-[1]^*]+2\,\text{Re}[M_+[1]\,S_-[1]^*]+\text{Re}[E_+[0]\,S_+[0]^*]+6\,\text{Re}[E_+[1]\,S_+[1]^*]-2\,\text{Re}[M_-[1]\,S_+[1]^*]+2\,\text{Re}[M_+[1]\,S_+[1]^*]))-\frac{3\sqrt{2}\,Q\,(\text{Re}[E_+[1]\,S_+[0]^*]+\text{Re}[M_+[1]\,S_+[0]^*]+2\,\text{Re}[E_+[0]\,S_+[1]^*])\,P_1[x]}{q}-\frac{12\sqrt{2}\,Q\,(\text{Re}[E_+[1]\,S_+[1]^*]+\text{Re}[M_+[1]\,S_+[1]^*])\,P_2[x]}{q}\right)$$

RtoLegendre[sp, R_{LTh}[S], 0]

$$\frac{1}{q}(\sqrt{2}\,Q\,(\text{Re}[E_+[0]\,S_-[1]^*]+\text{Re}[E_+[1]\,S_+[0]^*]-\text{Re}[M_-[1]\,S_+[0]^*]-\text{Re}[M_+[1]\,S_+[0]^*]))+\frac{1}{5\,q}(\sqrt{2}\,Q\,(15\,\text{Re}[E_+[1]\,S_-[1]^*]-5\,\text{Re}[M_-[1]\,S_-[1]^*]+5\,\text{Re}[M_+[1]\,S_-[1]^*]+5\,\text{Re}[E_+[0]\,S_+[0]^*]+24\,\text{Re}[E_+[1]\,S_+[1]^*]-20\,\text{Re}[M_-[1]\,S_+[1]^*]-16\,\text{Re}[M_+[1]\,S_+[1]^*])\,P_1[x])+\frac{2\sqrt{2}\,Q\,(\text{Re}[E_+[1]\,S_+[0]^*]+\text{Re}[M_+[1]\,S_+[0]^*]+2\,\text{Re}[E_+[0]\,S_+[1]^*])\,P_2[x]}{q}+\frac{36\sqrt{2}\,Q\,(\text{Re}[E_+[1]\,S_+[1]^*]+\text{Re}[M_+[1]\,S_+[1]^*])\,P_3[x]}{5\,q}$$

RtoLegendre[sp, R_{TTh}[L], 0]

$$2\,\text{Re}[M_-[1]\,E_+[0]^*]+\left(-\text{Abs}[E_+[0]]^2-\frac{9}{5}\text{Abs}[E_+[1]]^2-\text{Abs}[M_-[1]]^2-\text{Abs}[M_+[1]]^2+6\,\text{Re}[M_-[1]\,E_+[1]^*]+\frac{6}{5}\text{Re}[M_+[1]\,E_+[1]^*]+2\,\text{Re}[M_+[1]\,M_-[1]^*]\right)P_1[x]-2\,(\text{Re}[M_+[1]\,E_+[0]^*]+3\,\text{Re}[E_+[0]\,E_+[1]^*])\,P_2[x]-\frac{36}{5}(\text{Abs}[E_+[1]]^2+\text{Re}[M_+[1]\,E_+[1]^*])\,P_3[x]$$

RtoLegendre[sp, R_{TTh}[S], 1]

$$\text{Sin}[\theta]\,(-\text{Abs}[E_+[0]]^2-6\,\text{Abs}[E_+[1]]^2+\text{Abs}[M_-[1]]^2-2\,\text{Abs}[M_+[1]]^2+3\,\text{Re}[M_-[1]\,E_+[1]^*]+\text{Re}[M_+[1]\,M_-[1]^*]-3\,(\text{Re}[M_+[1]\,E_+[0]^*]+3\,\text{Re}[E_+[0]\,E_+[1]^*])\,P_1[x]-12\,(\text{Abs}[E_+[1]]^2+\text{Re}[M_+[1]\,E_+[1]^*])\,P_2[x])$$

RtoLegendre[sp, R_L[N], 1]

$$\text{Sin}[\theta]\left(\frac{2\,Q^2\,(\text{Im}[S_-[1]^*\,S_+[0]]-2\,\text{Im}[S_+[1]^*\,S_+[0]])}{q^2}+\frac{12\,Q^2\,\text{Im}[S_-[1]^*\,S_+[1]]\,P_1[x]}{q^2}\right)$$

RtoLegendre[sp, R_T[N], 1]

$$\mathrm{Sin}[\theta]\;(2\,\mathrm{Im}[M_-[1]\,E_+[0]^*] + \mathrm{Im}[M_+[1]\,E_+[0]^*] - 3\,\mathrm{Im}[E_+[0]\,E_+[1]^*] + (9\,\mathrm{Im}[M_-[1]\,E_+[1]^*] - 3\,\mathrm{Im}[M_+[1]\,M_-[1]^*])\,P_1[x])$$

RtoLegendre[sp, R_{LT}[N], 0]

$$\frac{1}{q}(\sqrt{2}\,Q\,(-\mathrm{Im}[E_+[1]\,S_-[1]^*] - \mathrm{Im}[M_-[1]\,S_-[1]^*] + \mathrm{Im}[M_+[1]\,S_-[1]^*] + \mathrm{Im}[E_+[0]\,S_+[0]^*] + 8\,\mathrm{Im}[E_+[1]\,S_+[1]^*])) + \frac{1}{q}(\sqrt{2}\,Q\,(\mathrm{Im}[E_+[0]\,S_-[1]^*] + 3\,\mathrm{Im}[E_+[1]\,S_+[0]^*] - \mathrm{Im}[M_-[1]\,S_+[0]^*] + \mathrm{Im}[M_+[1]\,S_+[0]^*] + 4\,\mathrm{Im}[E_+[0]\,S_+[1]^*])\,P_1[x]) + \frac{1}{q}(4\sqrt{2}\,Q\,(\mathrm{Im}[E_+[1]\,S_-[1]^*] + \mathrm{Im}[E_+[1]\,S_+[1]^*] - \mathrm{Im}[M_-[1]\,S_+[1]^*] + \mathrm{Im}[M_+[1]\,S_+[1]^*])\,P_2[x])$$

RtoLegendre[sp, R_{LT}[L], 1]

$$\mathrm{Sin}[\theta]\left(\frac{1}{q}(\sqrt{2}\,Q\,(\mathrm{Im}[M_-[1]\,S_-[1]^*] + 2\,\mathrm{Im}[M_+[1]\,S_-[1]^*] + \mathrm{Im}[E_+[0]\,S_+[0]^*] + 6\,\mathrm{Im}[E_+[1]\,S_+[1]^*] - 2\,\mathrm{Im}[M_-[1]\,S_+[1]^*] + 2\,\mathrm{Im}[M_+[1]\,S_+[1]^*])) + \frac{3\sqrt{2}\,Q\,(\mathrm{Im}[E_+[1]\,S_+[0]^*] + \mathrm{Im}[M_+[1]\,S_+[0]^*] + 2\,\mathrm{Im}[E_+[0]\,S_+[1]^*])\,P_1[x]}{q} + \frac{12\sqrt{2}\,Q\,(\mathrm{Im}[E_+[1]\,S_+[1]^*] + \mathrm{Im}[M_+[1]\,S_+[1]^*])\,P_2[x]}{q}\right)$$

RtoLegendre[sp, R_{LT}[S], 0]

$$-\frac{1}{q}(\sqrt{2}\,Q\,(\mathrm{Im}[E_+[0]\,S_-[1]^*] + \mathrm{Im}[E_+[1]\,S_+[0]^*] - \mathrm{Im}[M_-[1]\,S_+[0]^*] - \mathrm{Im}[M_+[1]\,S_+[0]^*])) - \frac{1}{5\,q}(\sqrt{2}\,Q\,(15\,\mathrm{Im}[E_+[1]\,S_-[1]^*] - 5\,\mathrm{Im}[M_-[1]\,S_-[1]^*] + 5\,\mathrm{Im}[M_+[1]\,S_-[1]^*] + 5\,\mathrm{Im}[E_+[0]\,S_+[0]^*] + 24\,\mathrm{Im}[E_+[1]\,S_+[1]^*] - 20\,\mathrm{Im}[M_-[1]\,S_+[1]^*] - 16\,\mathrm{Im}[M_+[1]\,S_+[1]^*])\,P_1[x]) - \frac{2\sqrt{2}\,Q\,(\mathrm{Im}[E_+[1]\,S_+[0]^*] + \mathrm{Im}[M_+[1]\,S_+[0]^*] + 2\,\mathrm{Im}[E_+[0]\,S_+[1]^*])\,P_2[x]}{q} - \frac{36\sqrt{2}\,Q\,(\mathrm{Im}[E_+[1]\,S_+[1]^*] + \mathrm{Im}[M_+[1]\,S_+[1]^*])\,P_3[x]}{5\,q}$$

RtoLegendre[sp, R_{TT}[N], 1]

$$\mathrm{Sin}[\theta]\;(-3\,(\mathrm{Im}[M_+[1]\,E_+[0]^*] + \mathrm{Im}[E_+[0]\,E_+[1]^*]) + 3\,(\mathrm{Im}[M_-[1]\,E_+[1]^*] - 4\,\mathrm{Im}[M_+[1]\,E_+[1]^*] + \mathrm{Im}[M_+[1]\,M_-[1]^*])\,P_1[x])$$

RtoLegendre[sp, R_{TT}[L], 2]

$$\text{Sin}[\theta]^2 \left(-3\,(\text{Im}[M_+[1]\,E_+[0]^*] + \text{Im}[E_+[0]\,E_+[1]^*]) - 18\,\text{Im}[M_+[1]\,E_+[1]^*]\,P_1[x]\right)$$

RtoLegendre[sp, R_{TT}[S], 1]

$$\text{Sin}[\theta]\,(-3\,(\text{Im}[M_-[1]\,E_+[1]^*] + \text{Im}[M_+[1]\,M_-[1]^*]) + 3\,(\text{Im}[M_+[1]\,E_+[0]^*] + \text{Im}[E_+[0]\,E_+[1]^*])\,P_1[x] + 12\,\text{Im}[M_+[1]\,E_+[1]^*]\,P_2[x])$$

■ P_{33}

RtoLegendre[P_{33}, R_L[0], 0]

$$\frac{8\,Q^2\,\text{Abs}[S_+[1]]^2}{q^2} + \frac{8\,Q^2\,\text{Re}[S_+[1]^*\,S_+[0]]\,P_1[x]}{q^2} + \frac{8\,Q^2\,(\text{Abs}[S_+[1]]^2 + \text{Re}[S_-[1]^*\,S_+[1]])\,P_2[x]}{q^2}$$

RtoLegendre[P_{33}, R_T[0], 0]

$$2\,(3\,\text{Abs}[E_+[1]]^2 + \text{Abs}[M_+[1]]^2) + 2\,(\text{Re}[M_+[1]\,E_+[0]^*] + 3\,\text{Re}[E_+[0]\,E_+[1]^*])\,P_1[x] + (3\,\text{Abs}[E_+[1]]^2 - \text{Abs}[M_+[1]]^2 - 6\,\text{Re}[M_-[1]\,E_+[1]^*] + 6\,\text{Re}[M_+[1]\,E_+[1]^*] - 2\,\text{Re}[M_+[1]\,M_-[1]^*])\,P_2[x]$$

RtoLegendre[P_{33}, R_{TT}[0], 2] // Collect[#, {Sin[θ], $P_{λ\_}$[x]}] &

$$\frac{3}{2}\,(3\,\text{Abs}[E_+[1]]^2 - \text{Abs}[M_+[1]]^2 - 2\,(-\text{Re}[M_-[1]\,E_+[1]^*] + \text{Re}[M_+[1]\,E_+[1]^*] + \text{Re}[M_+[1]\,M_-[1]^*]))\,\text{Sin}[\theta]^2$$

RtoLegendre[P_{33}, R_{LT}[0], 1]

$$\text{Sin}[\theta]\left(\frac{\sqrt{2}\,Q\,(-3\,\text{Re}[E_+[1]\,S_+[0]^*] + \text{Re}[M_+[1]\,S_+[0]^*] + 2\,\text{Re}[E_+[0]\,S_+[1]^*])}{q} - \frac{1}{q}(6\sqrt{2}\,Q\,(\text{Re}[E_+[1]\,S_-[1]^*] + \text{Re}[E_+[1]\,S_+[1]^*] + \text{Re}[M_-[1]\,S_+[1]^*] - \text{Re}[M_+[1]\,S_+[1]^*])\,P_1[x])\right)$$

RtoLegendre[P_{33}, R_{LTh}[0], 1]

$$\text{Sin}[\theta]\left(\frac{\sqrt{2}\,Q\,(3\,\text{Im}[E_+[1]\,S_+[0]^*] - \text{Im}[M_+[1]\,S_+[0]^*] - 2\,\text{Im}[E_+[0]\,S_+[1]^*])}{q} + \frac{1}{q}(6\sqrt{2}\,Q\,(\text{Im}[E_+[1]\,S_-[1]^*] + \text{Im}[E_+[1]\,S_+[1]^*] + \text{Im}[M_-[1]\,S_+[1]^*] - \text{Im}[M_+[1]\,S_+[1]^*])\,P_1[x])\right)$$

RtoLegendre[P_{33}, R_{LTh}[N], 0]

$$\frac{\sqrt{2}\,Q\,(-\text{Re}[E_+[1]\,S_-[1]^*] + \text{Re}[M_+[1]\,S_-[1]^*] + 8\,\text{Re}[E_+[1]\,S_+[1]^*])}{q} + \frac{\sqrt{2}\,Q\,(3\,\text{Re}[E_+[1]\,S_+[0]^*] + \text{Re}[M_+[1]\,S_+[0]^*] + 4\,\text{Re}[E_+[0]\,S_+[1]^*])\,P_1[x]}{q} + \frac{1}{q}(4\sqrt{2}\,Q\,(\text{Re}[E_+[1]\,S_-[1]^*] + \text{Re}[E_+[1]\,S_+[1]^*] - \text{Re}[M_-[1]\,S_+[1]^*] + \text{Re}[M_+[1]\,S_+[1]^*])\,P_2[x])$$

RtoLegendre[P_{33}, R_{LTh}[L], 1]

$$\text{Sin}[\theta]\left(-\frac{1}{q}(2\sqrt{2}\,Q\,(\text{Re}[M_+[1]\,S_-[1]^*] + 3\,\text{Re}[E_+[1]\,S_+[1]^*] - \text{Re}[M_-[1]\,S_+[1]^*] + \text{Re}[M_+[1]\,S_+[1]^*])) - \frac{3\sqrt{2}\,Q\,(\text{Re}[E_+[1]\,S_+[0]^*] + \text{Re}[M_+[1]\,S_+[0]^*] + 2\,\text{Re}[E_+[0]\,S_+[1]^*])\,P_1[x]}{q} - \frac{12\sqrt{2}\,Q\,(\text{Re}[E_+[1]\,S_+[1]^*] + \text{Re}[M_+[1]\,S_+[1]^*])\,P_2[x]}{q}\right)$$

RtoLegendre[P_{33}, R_{LTh}[S], 0]

$$\frac{\sqrt{2}\,Q\,(\text{Re}[E_+[1]\,S_+[0]^*] - \text{Re}[M_+[1]\,S_+[0]^*])}{q} + \frac{1}{5\,q}(\sqrt{2}\,Q\,(15\,\text{Re}[E_+[1]\,S_-[1]^*] + 5\,\text{Re}[M_+[1]\,S_-[1]^*] + 4\,(6\,\text{Re}[E_+[1]\,S_+[1]^*] - 5\,\text{Re}[M_-[1]\,S_+[1]^*] - 4\,\text{Re}[M_+[1]\,S_+[1]^*]))\,P_1[x]) + \frac{2\sqrt{2}\,Q\,(\text{Re}[E_+[1]\,S_+[0]^*] + \text{Re}[M_+[1]\,S_+[0]^*] + 2\,\text{Re}[E_+[0]\,S_+[1]^*])\,P_2[x]}{q} + \frac{36\sqrt{2}\,Q\,(\text{Re}[E_+[1]\,S_+[1]^*] + \text{Re}[M_+[1]\,S_+[1]^*])\,P_3[x]}{5\,q}$$

RtoLegendre[P_{33}, R_{TTh}[L], 0]

$$\left(-\frac{9}{5}\text{Abs}[E_+[1]]^2 - \text{Abs}[M_+[1]]^2 + 6\,\text{Re}[M_-[1]\,E_+[1]^*] + \frac{6}{5}\text{Re}[M_+[1]\,E_+[1]^*] + 2\,\text{Re}[M_+[1]\,M_-[1]^*]\right) P_1[x] - 2\,(\text{Re}[M_+[1]\,E_+[0]^*] + 3\,\text{Re}[E_+[0]\,E_+[1]^*])\,P_2[x] - \frac{36}{5}(\text{Abs}[E_+[1]]^2 + \text{Re}[M_+[1]\,E_+[1]^*])\,P_3[x]$$

RtoLegendre[P_{33}, R_{TTh}[S], 1]

$$\text{Sin}[\theta]\,(-6\,\text{Abs}[E_+[1]]^2 - 2\,\text{Abs}[M_+[1]]^2 + 3\,\text{Re}[M_-[1]\,E_+[1]^*] + \text{Re}[M_+[1]\,M_-[1]^*] - 3\,(\text{Re}[M_+[1]\,E_+[0]^*] + 3\,\text{Re}[E_+[0]\,E_+[1]^*])\,P_1[x] - 12\,(\text{Abs}[E_+[1]]^2 + \text{Re}[M_+[1]\,E_+[1]^*])\,P_2[x])$$

RtoLegendre[P_{33}, R_L[N], 1]

$$\text{Sin}[\theta]\left(-\frac{4\,Q^2\,\text{Im}[S_+[1]^*\,S_+[0]]}{q^2} + \frac{12\,Q^2\,\text{Im}[S_-[1]^*\,S_+[1]]\,P_1[x]}{q^2}\right)$$

RtoLegendre[P_{33}, R_T[N], 1]

$$\text{Sin}[\theta]\,(\text{Im}[M_+[1]\,E_+[0]^*] - 3\,\text{Im}[E_+[0]\,E_+[1]^*] + (9\,\text{Im}[M_-[1]\,E_+[1]^*] - 3\,\text{Im}[M_+[1]\,M_-[1]^*])\,P_1[x])$$

RtoLegendre[P_{33}, R_{LT}[N], 0]

$$\frac{\sqrt{2}\,Q\,(-\text{Im}[E_+[1]\,S_-[1]^*] + \text{Im}[M_+[1]\,S_-[1]^*] + 8\,\text{Im}[E_+[1]\,S_+[1]^*])}{q} + \frac{\sqrt{2}\,Q\,(3\,\text{Im}[E_+[1]\,S_+[0]^*] + \text{Im}[M_+[1]\,S_+[0]^*] + 4\,\text{Im}[E_+[0]\,S_+[1]^*])\,P_1[x]}{q} + \frac{1}{q}(4\sqrt{2}\,Q\,(\text{Im}[E_+[1]\,S_-[1]^*] + \text{Im}[E_+[1]\,S_+[1]^*] - \text{Im}[M_-[1]\,S_+[1]^*] + \text{Im}[M_+[1]\,S_+[1]^*])\,P_2[x])$$

RtoLegendre[P_{33}, R_{LT}[L], 1]

$$\text{Sin}[\theta]\left(\frac{1}{q}(2\sqrt{2}\,Q\,(\text{Im}[M_+[1]\,S_-[1]^*] + 3\,\text{Im}[E_+[1]\,S_+[1]^*] - \text{Im}[M_-[1]\,S_+[1]^*] + \text{Im}[M_+[1]\,S_+[1]^*])) + \frac{3\sqrt{2}\,Q\,(\text{Im}[E_+[1]\,S_+[0]^*] + \text{Im}[M_+[1]\,S_+[0]^*] + 2\,\text{Im}[E_+[0]\,S_+[1]^*])\,P_1[x]}{q} + \frac{12\sqrt{2}\,Q\,(\text{Im}[E_+[1]\,S_+[1]^*] + \text{Im}[M_+[1]\,S_+[1]^*])\,P_2[x]}{q}\right)$$

RtoLegendre[P_{33}, R_{LT}[S], 0]

$$-\frac{\sqrt{2}\,Q\,(\text{Im}[E_+[1]\,S_+[0]^*]-\text{Im}[M_+[1]\,S_+[0]^*])}{q}-\frac{1}{5\,q}(\sqrt{2}\,Q\,(15\,\text{Im}[E_+[1]\,S_-[1]^*]+5\,\text{Im}[M_+[1]\,S_-[1]^*]+4\,(6\,\text{Im}[E_+[1]\,S_+[1]^*]-5\,\text{Im}[M_-[1]\,S_+[1]^*]-4\,\text{Im}[M_+[1]\,S_+[1]^*]))\,P_1[x])-\frac{2\sqrt{2}\,Q\,(\text{Im}[E_+[1]\,S_+[0]^*]+\text{Im}[M_+[1]\,S_+[0]^*]+2\,\text{Im}[E_+[0]\,S_+[1]^*])\,P_2[x]}{q}-\frac{36\sqrt{2}\,Q\,(\text{Im}[E_+[1]\,S_+[1]^*]+\text{Im}[M_+[1]\,S_+[1]^*])\,P_3[x]}{5\,q}$$

RtoLegendre[P_{33}, R_{TT}[N], 1]

$$\text{Sin}[\theta]\,(-3\,(\text{Im}[M_+[1]\,E_+[0]^*]+\text{Im}[E_+[0]\,E_+[1]^*])+3\,(\text{Im}[M_-[1]\,E_+[1]^*]-4\,\text{Im}[M_+[1]\,E_+[1]^*]+\text{Im}[M_+[1]\,M_-[1]^*])\,P_1[x])$$

RtoLegendre[P_{33}, R_{TT}[L], 2]

$$\text{Sin}[\theta]^2\,(-3\,(\text{Im}[M_+[1]\,E_+[0]^*]+\text{Im}[E_+[0]\,E_+[1]^*])-18\,\text{Im}[M_+[1]\,E_+[1]^*]\,P_1[x])$$

RtoLegendre[P_{33}, R_{TT}[S], 1]

$$\text{Sin}[\theta]\,(-3\,(\text{Im}[M_-[1]\,E_+[1]^*]+\text{Im}[M_+[1]\,M_-[1]^*])+3\,(\text{Im}[M_+[1]\,E_+[0]^*]+\text{Im}[E_+[0]\,E_+[1]^*])\,P_1[x]+12\,\text{Im}[M_+[1]\,E_+[1]^*]\,P_2[x])$$

■ M1

RtoLegendre[M1, R_L[0], 0]

0

RtoLegendre[M1, R_T[0], 0]

$$2\,\text{Abs}[M_+[1]]^2+2\,\text{Re}[M_+[1]\,E_+[0]^*]\,P_1[x]+(-\text{Abs}[M_+[1]]^2+6\,\text{Re}[M_+[1]\,E_+[1]^*]-2\,\text{Re}[M_+[1]\,M_-[1]^*])\,P_2[x]$$

RtoLegendre[M1, R_{TT}[0], 2]

$$-\frac{3}{2}\,(\text{Abs}[M_+[1]]^2+2\,(\text{Re}[M_+[1]\,E_+[1]^*]+\text{Re}[M_+[1]\,M_-[1]^*]))\,\text{Sin}[\theta]^2$$

RtoLegendre[M1, R_{LT}[0], 1]

$$\text{Sin}[\theta]\left(\frac{\sqrt{2}\ Q\ \text{Re}[M_+[1]\ S_+[0]^*]}{q}+\frac{6\sqrt{2}\ Q\ \text{Re}[M_+[1]\ S_+[1]^*]\ P_1[x]}{q}\right)$$

RtoLegendre[M1, R_{LTh}[0], 1]

$$\text{Sin}[\theta]\left(-\frac{\sqrt{2}\ Q\ \text{Im}[M_+[1]\ S_+[0]^*]}{q}-\frac{6\sqrt{2}\ Q\ \text{Im}[M_+[1]\ S_+[1]^*]\ P_1[x]}{q}\right)$$

RtoLegendre[M1, R_{LTh}[N], 0]

$$\frac{\sqrt{2}\ Q\ \text{Re}[M_+[1]\ S_-[1]^*]}{q}+\frac{\sqrt{2}\ Q\ \text{Re}[M_+[1]\ S_+[0]^*]\ P_1[x]}{q}+\frac{4\sqrt{2}\ Q\ \text{Re}[M_+[1]\ S_+[1]^*]\ P_2[x]}{q}$$

RtoLegendre[M1, R_{LTh}[L], 1]

$$\text{Sin}[\theta]\left(-\frac{2\sqrt{2}\ Q\ (\text{Re}[M_+[1]\ S_-[1]^*]+\text{Re}[M_+[1]\ S_+[1]^*])}{q}-\frac{3\sqrt{2}\ Q\ \text{Re}[M_+[1]\ S_+[0]^*]\ P_1[x]}{q}-\frac{12\sqrt{2}\ Q\ \text{Re}[M_+[1]\ S_+[1]^*]\ P_2[x]}{q}\right)$$

RtoLegendre[M1, R_{LTh}[S], 0]

$$-\frac{\sqrt{2}\ Q\ \text{Re}[M_+[1]\ S_+[0]^*]}{q}+\frac{\sqrt{2}\ Q\ (5\ \text{Re}[M_+[1]\ S_-[1]^*]-16\ \text{Re}[M_+[1]\ S_+[1]^*])\ P_1[x]}{5\ q}+\frac{2\sqrt{2}\ Q\ \text{Re}[M_+[1]\ S_+[0]^*]\ P_2[x]}{q}+\frac{36\sqrt{2}\ Q\ \text{Re}[M_+[1]\ S_+[1]^*]\ P_3[x]}{5\ q}$$

RtoLegendre[M1, R_{TTh}[L], 0]

$$\left(-\text{Abs}[M_+[1]]^2+\frac{6}{5}\ \text{Re}[M_+[1]\ E_+[1]^*]+2\ \text{Re}[M_+[1]\ M_-[1]^*]\right)P_1[x]-2\ \text{Re}[M_+[1]\ E_+[0]^*]\ P_2[x]-\frac{36}{5}\ \text{Re}[M_+[1]\ E_+[1]^*]\ P_3[x]$$

RtoLegendre[M1, R_{TTh}[S], 1]

$$\text{Sin}[\theta]\ (-2\ \text{Abs}[M_+[1]]^2+\text{Re}[M_+[1]\ M_-[1]^*]-3\ \text{Re}[M_+[1]\ E_+[0]^*]\ P_1[x]-12\ \text{Re}[M_+[1]\ E_+[1]^*]\ P_2[x])$$

RtoLegendre[M1, R_L[N], 1]

0

RtoLegendre[M1, R_T[N], 1]

$$\text{Sin}[\theta]\ (\text{Im}[M_+[1]\ E_+[0]^*] - 3\ \text{Im}[M_+[1]\ M_-[1]^*]\ P_1[x])$$

RtoLegendre[M1, R_{LT}[N], 0]

$$\frac{\sqrt{2}\ Q\ \text{Im}[M_+[1]\ S_-[1]^*]}{q} + \frac{\sqrt{2}\ Q\ \text{Im}[M_+[1]\ S_+[0]^*]\ P_1[x]}{q} + \frac{4\sqrt{2}\ Q\ \text{Im}[M_+[1]\ S_+[1]^*]\ P_2[x]}{q}$$

RtoLegendre[M1, R_{LT}[L], 1]

$$\text{Sin}[\theta]\left(\frac{2\sqrt{2}\ Q\ (\text{Im}[M_+[1]\ S_-[1]^*] + \text{Im}[M_+[1]\ S_+[1]^*])}{q} + \frac{3\sqrt{2}\ Q\ \text{Im}[M_+[1]\ S_+[0]^*]\ P_1[x]}{q} + \frac{12\sqrt{2}\ Q\ \text{Im}[M_+[1]\ S_+[1]^*]\ P_2[x]}{q}\right)$$

RtoLegendre[M1, R_{LT}[S], 0]

$$\frac{\sqrt{2}\ Q\ \text{Im}[M_+[1]\ S_+[0]^*]}{q} + \frac{\sqrt{2}\ Q\ (-5\ \text{Im}[M_+[1]\ S_-[1]^*] + 16\ \text{Im}[M_+[1]\ S_+[1]^*])\ P_1[x]}{5\ q} - \frac{2\sqrt{2}\ Q\ \text{Im}[M_+[1]\ S_+[0]^*]\ P_2[x]}{q} - \frac{36\sqrt{2}\ Q\ \text{Im}[M_+[1]\ S_+[1]^*]\ P_3[x]}{5\ q}$$

RtoLegendre[M1, R_{TT}[N], 1]

$$\text{Sin}[\theta]\ (-3\ \text{Im}[M_+[1]\ E_+[0]^*] + 3\ (-4\ \text{Im}[M_+[1]\ E_+[1]^*] + \text{Im}[M_+[1]\ M_-[1]^*])\ P_1[x])$$

RtoLegendre[M1, R_{TT}[L], 2]

$$\text{Sin}[\theta]^2\ (-3\ \text{Im}[M_+[1]\ E_+[0]^*] - 18\ \text{Im}[M_+[1]\ E_+[1]^*]\ P_1[x])$$

RtoLegendre[M1, R_{TT}[S], 1]

$$\text{Sin}[\theta]\ (-3\ \text{Im}[M_+[1]\ M_-[1]^*] + 3\ \text{Im}[M_+[1]\ E_+[0]^*]\ P_1[x] + 12\ \text{Im}[M_+[1]\ E_+[1]^*]\ P_2[x])$$

Compare multipole expansions with Lourie proposal

I recently became aware of many significant discrepancies between the results of the UMd code for electroproduction of pseudoscalar mesons, *epiprod*, and those presented by Robert Lourie in both proposals and publications. Of most immediate concern are the discrepancies concerning CEBAF proposal 91-011, where Lourie quotes multipole expansions of recoil-polarization response functions, based upon those given by Raskin and Donnelly, assuming M_{1+} dominance. He also shows figures for many of the angular distributions that look qualitatively consistent with the corresponding multipole expansions. However, the results that I obtain using *epiprod* are often quite different, both algebraically and numerically. For example, the figure he gives for $R_{\text{TTh}}[L]$ is consistent with the 3 - Cos[2θ] shape given by his expression, while my figure has a Cos[θ] distribution that is consistent with the results above. When I checked a few special cases, I found that the UMd multipole expansions were often quite different from those of Raskin and Donnelly, but consistent with those of Drechsel and Tiator from Mainz. On the other hand, Donnelly expressed considerable confidence in his expressions and questioned the reliability of the Mainz work. Therefore, it became incumbent upon me to resolve this discrepancy.

There appear to be two important errors in the Raskin and Donnelly paper. First, they claim that the ejectile basis is used for recoil polarization but from the comparison below I conclude that they actually used the target basis. Second, their expressions appear to employ the ejectile angle but I find it necessary to replace that angle with the pion angle instead. [Note that because θ is the pion angle in the derivation above, I should have had to replace Cos[θ] → −Cos[θ] in order to obtain the correct relative sign between even and odd multipoles, but that replacement was not necessary because they actually used the pion angle.] To demonstrate these claims, I reproduce the helicity-dependent response functions for polarization components in the reaction plane given by Lourie. [Also note that Louries formula for $R_{\text{TTh}}[L]$ differs from Raskin and Donnelly in the sign of the last term; that mistake is corrected here.]

```
Lourie[TTh, L] = Simplify /@
  ((( 5/3 Abs[M+[1]]^2 - 2 Re[E+[1]* M+[1] + 2/3 M-[1]* M+[1]] + 2 Re[E+[0]* M+[1]]
        P1[x] + (-2/3 Abs[M+[1]]^2 + Re[8 E+[1]* M+[1] - 2/3 M-[1]* M+[1]])
        P2[x] /. ExpandLegendre /. x → Cos[θ] // Simplify) //.
     ExpandMultipoleProducts // Collect[#, {Abs[_], Re[_], Im[_]}] &)
```

$$-\frac{1}{2} \text{Abs}[M_+[1]]^2 (-3 + \text{Cos}[2\theta]) + 2\,\text{Cos}[\theta]\,\text{Re}[M_+[1]\,E_+[0]^*] + 6\,\text{Cos}[2\theta]\,\text{Re}[M_+[1]\,E_+[1]^*] - \frac{1}{2}(3 + \text{Cos}[2\theta])\,\text{Re}[M_+[1]\,M_-[1]^*]$$

```
Lourie[TTh, S] =
 Simplify /@ ((Sin[θ] (Re[E+[0]* M+[1]] + (Abs[M+[1]]^2 + Re[6 E+[1]* M+[1] +
          M-[1]* M+[1]]) P1[x]) /. ExpandLegendre /.
      x → Cos[θ] // Simplify) //. ExpandMultipoleProducts //
    Collect[#, {Abs[_], Re[_], Im[_]}] &)
```

```
Abs[M+[1]]^2 Cos[θ] Sin[θ] + Re[M+[1] E+[0]*] Sin[θ] +
 6 Cos[θ] Re[M+[1] E+[1]*] Sin[θ] + Cos[θ] Re[M+[1] M-[1]*] Sin[θ]
```

```
Lourie[LTh, L] = Simplify /@
   ((-√2 Sin[θ] (Re[2 S+[0]* M+[1] + (S-[1]* M+[1] + 10 S+[1]* M+[1]) P1[
           x]]) /. ExpandLegendre /. x → Cos[θ] // Simplify) //.
     ExpandMultipoleProducts // Collect[#, {Abs[_], Re[_], Im[_]}] &);
```

```
Lourie[LTh, S] =
  Simplify /@ ((√2 Re[5/3 S-[1]* M+[1] - 4/3 S+[1]* M+[1] + S+[0]* M+[1] P1[x] -
           2/3 (S-[1]* M+[1] - 8 S+[1]* M+[1]) P2[x]] /. ExpandLegendre /.
        x → Cos[θ] // Simplify) //. ExpandMultipoleProducts //
     Collect[#, {Abs[_], Re[_], Im[_]}] &);
```

Applying a rotation to the Lourie formulas and inserting the relative normalizations between our definitions of the interference response functions, we can now obtain results that are consistent with the present derivation.

```
(roty[-θ].{Lourie[LTh, S], 0, Lourie[LTh, L] } ==
   ((√2 AbbreviateMultipoleExpansion[#, M+[1]]) & /@
     {mpR_LTh[S], 0, mpR_LTh[L]})) // Simplify
```

```
{1/√2 (2 Cos[θ] Re[M+[1] S-[1]*] + (-1 + 3 Cos[2 θ]) Re[M+[1] S+[0]*] +
     (-Cos[θ] + 9 Cos[3 θ]) Re[M+[1] S+[1]*]), 0,
  -√2 (2 Re[M+[1] S-[1]*] + 3 Cos[θ] Re[M+[1] S+[0]*] +
      (5 + 9 Cos[2 θ]) Re[M+[1] S+[1]*]) Sin[θ]} ==
 {1/q (2 Q (x Re[M+[1] S-[1]*] + (-2 + 3 x^2) Re[M+[1] S+[0]*] +
       2 x (-7 + 9 x^2) Re[M+[1] S+[1]*])),
  0, -1/q (2 Q (2 Re[M+[1] S-[1]*] + 3 x Re[M+[1] S+[0]*] +
        2 (-2 + 9 x^2) Re[M+[1] S+[1]*]) Sin[θ])}
```

```
(roty[-θ].{Lourie[TTh, S], 0, Lourie[TTh, L] } ==
   ((-AbbreviateMultipoleExpansion[#, M₊[1]]) & /@
     {mpR_TTh[S], 0, mpR_TTh[L]})) // Simplify
```

```
{3 (-x + Cos[θ]) (Re[M₊[1] E₊[0]*] + 6 (x + Cos[θ]) Re[M₊[1] E₊[1]*]) Sin[θ],
  0, (-x + Cos[θ])
   (Abs[M₊[1]]^2 + 3 (x + Cos[θ]) Re[M₊[1] E₊[0]*] - 3 Re[M₊[1] E₊[1]*] +
     18 x^2 Re[M₊[1] E₊[1]*] + 18 x Cos[θ] Re[M₊[1] E₊[1]*] +
     9 Cos[2 θ] Re[M₊[1] E₊[1]*] - 2 Re[M₊[1] M₋[1]*])} == {0, 0, 0}
```

Thus apart from trivial normalization factors we recover Lourie's multipole expansions of the helicity-dependent in-plane response functions by replacing θ_N by θ_π and rotating to the target basis. I have not checked the helicity-independent response functions, but expect the same behavior. Furthermore, I incidentally encountered remarks by Dmitrasinovic [Phys. Rev. 51, 1528 (1995), footnote 1] indicating that both he and Hanstein [Mainz Ph.D thesis] have also found that a rotation is needed to correct the Lourie expressions, although he does not mention the replacement of θ_N by θ_π. It turns out that Donnelly remembered a conversation with the Mainz group along similar lines, but thought that his results were correct and that Mainz had corrected their problem. However, unless I am seriouly misreading the Raskin and Donnelly paper, I have to conclude that the MIT results were represented incorrectly. Therefore, I will continue to trust *epiprod* and will disregard the expressions and figures in Lourie's proposal. I will also proceed to develop the target polarization using the present formalism and methods.