Comment on “Wigner-Thomas spin precession in polarized coincidence electronuclear scattering”

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(Dated: October 28, 2004)

A pedagogical paper by Dmitrašinović [Phys. Rev. C **47**, 2195 (1993)] discusses the cm $\rightarrow$ lab transformation of nucleon polarization and was intended to guide experimentalists in the analysis of data for $(\vec{e}, e'\vec{N})$ reactions. Unfortunately, his transformation employs the wrong angle and his description of the effect is misleading, which has caused confusion in some of those analyses. Here I attempt to remedy that situation.

PACS numbers: 25.30.-c,24.70.+s,13.60.-r

In 1993, Dmitrašinović [1] published a “frankly pedagogical paper” about the cm $\rightarrow$ lab transformation of nucleon spin and its effect upon analysis of response functions for recoil polarization in electron scattering reactions of the $(\vec{e}, e'\vec{N})$ type. Although the basic effect has been known for a long time (*e.g.* [2–4]), his paper is cited frequently by experimentalists because it provides explicit formulas for the transformation of response functions and a more detailed discussion of the nature and size of its effects. However, his transformation employs the wrong angle and his description of the effect is misleading, which has caused considerable confusion in the early stages of analyses of several experiments. Although I believe that the misconceptions were eventually dispelled and that no published data were adversely affected, a formal correction should help prevent future problems.

To clarify matters, it will help to review the standard derivation [2–4]. Consider two reference frames, S_1 and S_2, that share a common set of coordinate axes $(\hat{x}, \hat{y}, \hat{z})$ and differ by a boost along $\hat{z}$. We assume, without loss of generality, that the ejectile momentum lies in the $(\hat{x}, \hat{z})$ plane. If the momentum $\vec{p}_1$ forms an angle θ_1 with the $\hat{z}$ axis of S_1, a rest frame labelled R_1 with the same coordinate axes is reached by the transformation

$$T_1 = R_y(-\theta_1) B_z(\beta_1) R_y(\theta_1) \tag{1}$$

where R_y is a rotation about $\hat{y}$ and B_z is a boost along $\hat{z}$. Alternatively, we could first use a boost along the $\hat{z}$ axis to transform S_1 to S_2, in which the momentum $\vec{p}_2$ appears at angle θ_2, and then a boost along $\vec{\beta}_2$ to reach another rest frame labelled R_2. Thus, the transformation

$$T_2 = R_y(-\theta_2) B_z(\beta_2) R_y(\theta_2) B_z(-\beta_{1,2}) \tag{2}$$

produces rest frame R_2, where $\beta_{1,2}$ is the velocity of S_1 relative to S_2. These are both rest frames of the same particle, so the spin of the particle in frames S_1 and S_2 can differ at most by a rotation about their common $\hat{y}$ axis, such that

$$R_y(\chi) = T_2 T_1^{-1} \tag{3}$$

Using that fact that successive rotations about the same axis add, it is easier to evaluate

$$R_y(-\Omega) = B_z(\beta_2) R_y(\theta_2) B_z(-\beta_{1,2}) R_y(-\theta_1) B_z(-\beta_1) \tag{4}$$

where $\Omega = \theta_1 - \theta_2 - \chi$. Direct multiplication and simplification reveals that $T_2 T_1^{-1}$ is indeed a pure rotation and that

$$\tan\Omega = \frac{\beta_{1,2}\sin\theta_1}{\gamma_1(\beta_1 + \beta_{1,2}\cos\theta_1)} \tag{5a}$$

$$\tan\theta_2 = \frac{\beta_1\sin\theta_1}{\gamma_{1,2}(\beta_{1,2} + \beta_1\cos\theta_1)} \tag{5b}$$

$$\chi = \theta_1 - \theta_2 - \Omega \tag{5c}$$

These results are quite general and apply to any spin or to any pair of reference frames, although the most common applications are probably to the lab $\leftrightarrow$ cm transformations for spin one-half. Also note that $\beta_{1,2}$ may be positive or negative. The signs for χ and Ω were chosen to be positive when $\beta_{1,2} > 0$ as in the cm $\rightarrow$ lab transformation.

Recoil polarization observables are commonly expressed relative to coordinate systems with the third component along either the momentum transfer or the ejectile momentum. Let $\boldsymbol{q}$ represent the momentum transferred by electron scattering, $\boldsymbol{p}$ represent the momentum of a nucleon detected in coincidence, and $\boldsymbol{P}$ represent the recoil polarization for that nucleon. It is most convenient to represent the polarization in either a z-basis

$$\hat{z} = \hat{q},\ \hat{y} \propto \hat{q}\wedge\hat{p},\ \hat{x} = \hat{y}\wedge\hat{z} \tag{6}$$

with fixed orientation or helicity bases

$$\hat{\ell} = \hat{p},\ \hat{n} = \hat{y},\ \hat{t} = \hat{n}\wedge\hat{\ell} \tag{7}$$

whose orientation is tied to the ejectile momentum. The helicity basis is normally used when polarization observables are decomposed into electromagnetic response functions. Thus, one obtains

$$P_z^{(2)} = P_z^{(1)} \cos\chi + P_x^{(1)} \sin\chi \tag{8a}$$
$$P_x^{(2)} = -P_z^{(1)} \sin\chi + P_x^{(1)} \cos\chi \tag{8b}$$

in the z-basis or

$$P_\ell^{(2)} = P_\ell^{(1)} \cos\Omega - P_t^{(1)} \sin\Omega \tag{9a}$$
$$P_t^{(2)} = P_\ell^{(1)} \sin\Omega + P_t^{(1)} \cos\Omega \tag{9b}$$

in helicity bases. If S_1 is the cm frame and S_2 is the lab frame, then the relative velocity is expressed in terms of the lab momentum transfer (ω, q) and target mass M as $\beta_{1,2} = q/(M+\omega)$ and is positive, such that $\theta_2 < \theta_1$. One then finds that Ω is slightly less than $\theta_1 - \theta_2$ such that χ is a small positive angle with a maximum for $\theta_1 \sim 90^\circ$. Therefore, in the cm $\rightarrow$ lab transformation the polarization vector rotates forward, typically by a few degrees, with respect to a fixed basis and the mixing in Eq. (8) is small. On the other hand, the mixing in Eq. (9) with respect to the helicity basis can be quite large because the longitudinal direction is typically much more forward in the lab than in the cm frame such that the polarization appears to be rotated backward through a large angle with respect to a basis that rotates with the momentum.

Mosconi *et al.* [5] give the same results for Eq. (9). Arenhövel *et al.* [6] quote a formula from Giebink [4] for $\sin\Omega = \sin(\theta_1 - \theta_2)(1+\gamma_{1,2})/(\gamma_1+\gamma_2)$ that has the advantage that the nonrelativistic limit $\Omega \rightarrow \theta_1 - \theta_2$ is apparent and demonstrates that nonrelativistically there is no spin rotation relative to the z basis. However, in using that formula to obtain Ω one must be careful to recognize that $\beta > \beta_1$ permits $\cos\Omega$ to change sign when $\cos\theta_1 < -\beta_1/\beta$.

Equation (26) of Ref. [1], giving the transformation for response functions in the helicity basis, should have been equivalent to my Eq. (9) but it replaced Ω by $-\omega$ where

$$\tan\frac{\omega}{2} = \frac{\gamma_{1,2}\beta_{1,2}\gamma_1\beta_1 \sin\theta_1}{(1+\gamma_{1,2})(1+\gamma_1) + \gamma_{1,2}\beta_{1,2}\gamma_1\beta_1\cos\theta_1} \tag{10}$$

One can prove that $\omega = \chi$, but the algebra is rather tedious. His Fig. 3 indeed shows that ω is small, positive, and peaks near $\theta_1 \sim 90^\circ$ for a typical cm $\rightarrow$ lab transformation. However, according to Dmitrašinović, "ω is the precession angle with respect to the rotated direction of motion". Somewhat later he states: "The angle ω is the precession angle of the spin with respect to the coordinate system defined by the direction of motion of the particle, and will be referred to as the Wick-Wigner helicity precession angle." Thus, his Eq. (26) suggests that laboratory response functions are obtained from cm response functions using a rotation through an angle ω that produces relatively little mixing between longitudinal and transverse components. When examining the nonrelativistic limit, he claims "... there is no spin rotation relative to the momentum, i.e. no Wick-Wigner precession, in the nonrelativistic limit: The spin and the velocity three-vectors remain parallel in that limit." Apparently he claims that a polarization that is longitudinal in the cm appears to be longitudinal in the lab for nonrelativistic kinematics and nearly longitudinal relativistically. Near the end of the paper he states "... the spin direction essentially rotates together with the nucleon momentum" for deuteron electrodisintegration. Therefore, Dmitrašinović offers an unambiguous but incorrect interpretation of the Wigner transformation of polarization under a boost that is not parallel to a particle's momentum. These statements reveal that the error in his Eq. (26) is conceptual, not typographical.

Nonrelativistically the internal angular momentum of a system is invariant with respect to a Galilean transformation and does not rotate with respect to a basis with fixed orientation. Consequently, when the momentum rotates forward in a cm $\rightarrow$ lab transformation the spin would appear to rotate backward through the same angle if expressed relative to the helicity basis tied to the momentum. The additional relativistic rotation of spin arises because nonparallel Lorentz boosts do not commute; hence, χ is sometimes described as a *hyperbolic defect* [7] because addition of velocities is described by hyperbolic instead of Euclidean geometry. Nevertheless, the relativistic effect upon polarization is small for recent $(\vec{e}, e'\vec{N})$ experiments while the mixing between response functions for longitudinal and transverse recoil polarization is large simply because the momentum rotates much more than the polarization under a cm $\rightarrow$ lab transformation.

[1] V. Dmitrašinović, Phys. Rev. **C 47**, 2195 (1993).
[2] E. P. Wigner, Rev. Mod. Phys. **29**, 255 (1957).
[3] H. P. Stapp, Phys. Rev. **103**, 425 (1956).
[4] D. R. Giebink, Phys. Rev. **C 32**, 502 (1985).
[5] B. Mosconi, J. Pauschenwein, and P. Ricci, Phys. Rev. **C 48**, 332 (1993).
[6] H. Arenhövel, W. Leidemann, and E. L. Tomusiak, Few–Body Systems **15**, 109 (1993).
[7] G. C. Wick, Ann. Phys. (N.Y.) **18**, 65 (1962).