Accuracy of traditional Legendre estimators of quadrupole ratios for the $N \to \Delta$ transition

J. J. Kelly

Department of Physics, University of Maryland, College Park, Maryland 20742, USA

(Dated: July 6, 2005)

We evaluate the accuracy of traditional estimators often used to extract $N \to \Delta$ quadrupole ratios from cross section angular distributions for pion electroproduction. We find that neither M_{1+} dominance nor $\ell \leq 1$ truncation is sufficiently accurate for this purpose. Truncation errors are especially large for R_{EM}, for which it is also essential to perform Rosenbluth separation. The accuracy of similar truncated Legendre analyses for E_{0+}, S_{0+}, and especially M_{1-} is even worse.

PACS numbers: 14.20.Gk,13.60.Le,13.40.Gp,13.88.+e

Historically the most important indications of deformation of low-lying baryons have been the quadrupole ratios for electromagnetic excitation of the $N \to \Delta(1232)$ transition. Magnetic dipole excitation dominates and is represented by the M_{1+} multipole amplitude while nonzero values for the electric and scalar (longitudinal) multipoles, E_{1+} and S_{1+}, arise either from nonspherical contributions to the wave functions or from higher-order dynamical contributions to the electromagnetic transition. The quadrupole ratios are defined as

$$R_{EM} = \mathrm{Re}\frac{E_{1+}}{M_{1+}} \tag{1a}$$

$$R_{SM} = \mathrm{Re}\frac{S_{1+}}{M_{1+}} \tag{1b}$$

evaluated at the physical mass of the resonance, $W = M_\Delta \approx 1.232$ GeV. Determination of quadrupole ratios for isospin-3/2 amplitudes requires measurements of two charge states, such as $p\pi^0$ and $n\pi^+$. Complex multipole amplitudes have been deduced for $Q^2 = 0$ using polarization data for pion photoproduction [1], but few experiments for $Q^2 > 0$ have provided sufficient information to perform an actual multipole analysis. Instead, most experimental determinations of $N \to \Delta$ transition form factors [2–5] rely upon estimators derived from multipole expansions for the angular dependence of unpolarized cross sections using two simplifying assumptions: 1) only multipoles with $\ell \leq 1$ contribute, which is described as *sp* truncation; and 2) only terms involving M_{1+} are retained, which is described as M_{1+} dominance. Although the reliability of these assumptions has been questioned before, the improved kinematic completeness and statistical precision of modern experiments warrants re-examination of their accuracy. In this Brief Report, we consider the accuracy of traditional quadrupole estimators for $Q^2 \sim 1$ (GeV/c)^2 where a nearly model-independent multipole analysis of recoil-polarization response functions for the $p(\vec{e}, e'\vec{p})\pi^0$ reaction disagrees appreciably with the traditional Legendre analysis of the cross section data [6, 7].

The unpolarized differential cross section for $\gamma_v N \to N\pi$ in the πN center of momentum frame takes the form

$$\frac{d\sigma}{d\Omega_\pi} = \nu_0 \left(\epsilon_S R_L + R_T + \sqrt{2\epsilon_S(1+\epsilon)} R_{LT} \sin\theta\cos\phi + \epsilon R_{TT} \sin^2\theta \cos 2\phi \right) \tag{2}$$

where ν_0 is a phase-space factor, ϵ is the transverse polarization of the virtual photon, $\epsilon_S = \epsilon Q^2/q^2$, and (θ, ϕ) are polar and azimuthal pion angles relative to the $\vec{q}$ vector and the electron scattering plane. The response functions can be expanded in Legendre series

$$R_\lambda = \sum_{n=0}^{\infty} A_n^\lambda P_n(\cos\theta) \tag{3}$$

where $\lambda \in \{L, T, LT, TT\}$. The expansion coefficients, A_n^λ, are functions of (W, Q^2) that can be fit to the angular distribution of the differential cross section. Each of those coefficients can in turn be expressed as a multipole expansion containing terms of the form $\mathrm{Re}B_{\ell\pm}C^*_{\ell\pm}$ where $B, C \in \{M, E, S\}$ are magnetic, electric, or scalar multipole amplitudes for specified ℓ and $j = \ell \pm 1/2$. In principle, experimentally determined Legendre coefficients include contributions from arbitrarily large ℓ and are not limited either by *sp* truncation or by M_{1+} dominance.

Truncated multipole expansions of the Legendre coefficients used in quadrupole estimators are given in Eq. (4) where the contributions that satisfy M_{1+} dominance are listed first and where the remaining terms include some of the

TABLE I: Complexity of multipole expansions of Legendre coefficients. For each Legendre coefficient used for quadrupole estimators we show the number of independent terms of the form $\mathrm{Re}(ab^*)$, where a and b are multipole amplitudes with $\ell \leq \ell_{\max}$.

	complete						M_{1+} dominance					
$\ell_{\max}$	A_0^L	A_0^T	A_1^{LT}	A_0^{TT}	A_2^L	A_2^T	A_0^L	A_0^T	A_1^{LT}	A_0^{TT}	A_2^L	A_2^T
1	3	4	8	9	5	9	0	2	2	4	0	4
2	5	8	26	27	11	26	0	2	2	4	0	4
3	7	12	50	52	17	46	0	2	4	8	0	4
4	9	16	84	85	23	66	0	2	4	8	0	6
5	11	20	113	116	27	82	0	2	5	11	0	6
6	13	14	150	153	31	98	0	2	5	11	0	6

lowest multipolarity contributions of other types but are not necessarily arranged in order of numerical importance.

$$A_0^L = |S_{0+}|^2 + 8|S_{1+}|^2 + |S_{1-}|^2 + 8|S_{2-}|^2 + 27|S_{2+}|^2 + \ldots \tag{4a}$$

$$A_0^T = 2|M_{1+}|^2 + |E_{0+}|^2 + |M_{1-}|^2 + 6|E_{1+}|^2] + 6|M_{2-}|^2 + 2|E_{2-}|^2 + 9|M_{2+}|^2 + 18|E_{2+}|^2 + \ldots \tag{4b}$$

$$\begin{aligned} A_0^{TT} &= -\frac{3}{2}|M_{1+}|^2 - \mathrm{Re}[M_{1+}^*(3E_{1+} + 3M_{1-} + 12M_{3-} + 3E_{3-} + 2M_{3+} + 10E_{3+})] \\ &+ \frac{9}{2}|E_{1+}|^2 + \frac{3}{2}|E_{2-}|^2 + 24|E_{2+}|^2 - \frac{9}{2}|M_{2-}|^2 - 12|M_{2+}|^2 \\ &+ \mathrm{Re}[-3E_{0+}^*(E_{2-} + M_{2-} - M_{2+} + E_{2+}) + E_{1+}^*(3M_{1-} - 21E_{3-} - 12M_{3-} + 12M_{3+}) \\ &+ M_{1-}^*(3E_{3-} + 3M_{3-} + 10E_{3+} - 10M_{3+}) + \ldots] \end{aligned} \tag{4c}$$

$$\begin{aligned} A_1^{LT} &= 3\mathrm{Re}[M_{1+}^*(2S_{1+} - 3S_{3-} + 4S_{3+}) + S_{0+}^*(E_{2-} - M_{2-} + M_{2+} - 4E_{2+}) \\ &- E_{0+}^*(2S_{2-} - 3S_{2+}) - 2E_{1+}^*(S_{1+} + S_{1-}) - 2M_{1-}S_{1+}^* + \ldots] \end{aligned} \tag{4d}$$

$$A_2^L = 8|S_{1+}|^2 + 8|S_{2-}|^2 + \frac{216}{7}|S_{2+}|^2 + \mathrm{Re}[S_{0+}^*(8S_{2-} + 18S_{2+}) + 8S_{1+}S_{1-}^* + \ldots] \tag{4e}$$

$$\begin{aligned} A_2^T &= -|M_{1+}|^2 + \mathrm{Re}[M_{1+}^*(6E_{1+} - 2M_{1-} + \frac{24}{7}M_{3-} + 6E_{3-} + \frac{144}{7}M_{3+})] \\ &+ 3|E_{1+}|^2 - |E_{2-}|^2 + \frac{108}{7}|E_{2+}|^2 + 3|M_{2-}|^2 + \frac{36}{7}|M_{2+}|^2 \\ &+ \mathrm{Re}[E_{0+}^*(2E_{2-} - 6M_{2-} + 6M_{2+} + 12E_{2+}) - 6M_{1-}E_{1+}^* + \ldots] \end{aligned} \tag{4f}$$

Table I shows that the number of independent terms in the multipole expansions of these Legendre coefficients increases very rapidly with the maximum ℓ permitted. Complete expansions for $\ell_{\max} \leq 6$ can be found in Ref. [8] but as $\ell_{\max}$ increases they quickly become too unwieldy to display here or to use in practical applications. The Legendre coefficients are usually obtained by numerical integration of response functions against Legendre functions instead of by these algebraic formulas, but both methods do agree.

The assumption of M_{1+} dominance omits any terms that do not involve M_{1+}, which strongly inhibits the proliferation of terms but is not sufficient in itself to extract quadrupole ratios from cross section data. Combined with *sp* truncation, these expansions reduce to

$$A_0^L \approx 0 \tag{5a}$$

$$A_0^T \approx 2|M_{1+}|^2 \tag{5b}$$

$$A_0^{TT} \approx -\mathrm{Re}[(\frac{3}{2}M_{1+} + 3E_{1+} + 3M_{1-})M_{1+}^*] \tag{5c}$$

$$A_1^{LT} \approx 6\mathrm{Re}[S_{1+}M_{1+}^*] \tag{5d}$$

$$A_2^L \approx 0 \tag{5e}$$

$$A_2^T \approx \mathrm{Re}[(-M_{1+} + 6E_{1+} - 2M_{1-})M_{1+}^*] \tag{5f}$$

Thus, one obtains the traditional quadrupole estimators

$$\tilde{R}_{EM} = \frac{3(A_2^T + \epsilon A_2^L) - 2A_0^{TT}}{12(A_0^T + \epsilon A_0^L)} \tag{6a}$$

$$\tilde{R}_{SM} = \frac{A_1^{LT}}{3(A_0^T + \epsilon A_0^L)} \tag{6b}$$

TABLE II: Convergence of multipole expansions of Legendre coefficients and quadrupole estimators. Multipole amplitudes from MAID2003 for the $p\pi^0$ channel were used for $W = 1.232$ GeV and $Q^2 = 1.0$ (GeV/c)^2. Legendre coefficients are in units of $(\mu b)^{1/2}$.

							$\epsilon = 0.95$		$\epsilon = 0$	
$\ell_{\max}$	A_0^L	A_0^T	A_1^{LT}	A_0^{TT}	A_2^L	A_2^T	f_{EM}	f_{SM}	f_{EM}	f_{SM}
1	0.3339	7.599	-1.483	-5.220	0.141	-3.769	0.300	0.938	0.581	0.977
2	0.3377	7.624	-1.395	-5.156	0.102	-3.916	0.736	0.879	0.960	0.916
3	0.3384	7.628	-1.323	-5.137	0.079	-3.871	0.714	0.833	0.894	0.868
4	0.3384	7.628	-1.315	-5.130	0.081	-3.859	0.696	0.828	0.878	0.863
5	0.3384	7.628	-1.308	-5.130	0.081	-3.857	0.693	0.824	0.876	0.859

where A_n^L is included because most experiments have not used Rosenbluth separation to isolate A_n^T. When Rosenbluth separation is available, one can simply use $\epsilon \to 0$ in Eq. (6). Therefore, it is useful to define the accuracy parameters

$$f_{EM}(\ell_{\max}, \epsilon) = \frac{1}{R_{EM}} \left(\frac{3(A_2^T + \epsilon A_2^L) - 2A_0^{TT}}{12(A_0^T + \epsilon A_0^L)} \right)_{\ell \le \ell_{\max}} \simeq \frac{\tilde{R}_{EM}}{R_{EM}} \tag{7a}$$

$$f_{SM}(\ell_{\max}, \epsilon) = \frac{1}{R_{SM}} \left(\frac{A_1^{LT}}{3(A_0^T + \epsilon A_0^L)} \right)_{\ell \le \ell_{\max}} \simeq \frac{\tilde{R}_{SM}}{R_{SM}} \tag{7b}$$

where asymptotic equality refers to the limit $\ell_{\max} \to \infty$. Despite their appealing simplicity, it is clear that many contributions are omitted and the accuracy of the traditional estimators is a numerical issue that can be addressed either theoretically using model calculations or experimentally using additional polarization measurements to extract complex multipole amplitudes directly.

The convergence of these expansions is evaluated in Table II using $p\pi^0$ multipole amplitudes for $W = 1.232$ GeV and $Q^2 = 1.0$ (GeV/c)^2 from MAID2003 [9, 10]. First, we observe that A_n^L contributions are not negligible: the contribution of A_0^L to the denominators of Eq. (6) reduces the estimated quadrupole ratios by about 4% without Rosenbluth separation when $\epsilon \to 1$. (Note that $\epsilon = 0.949$ at $W = 1.232$ GeV in Ref. [6].) Even though A_2^L is much smaller, its effect upon f_{EM} is even larger because the strong cancellation between A_0^{TT} and $A_2^T + \epsilon A_2^L$ amplifies the dependence on ϵ. Therefore, the assumption of M_{1+} dominance is not sufficiently accurate to measure R_{EM} without Rosenbluth separation. Even with Rosenbluth separation, one should not expect better than about 15% accuracy for either quadrupole ratio using the traditional Legendre analysis (see the bottom of last two columns of Table II). Second, it is clear that *sp* truncation is not valid either because contributions with $\ell > 1$ are not negligible. Cancellation between contributions to the numerator of $\tilde{R}_{EM}$ also amplifies truncation errors and convergence is not necessarily monotonic as $\ell_{\max}$ increases. Contributions to A_0^L and A_0^T are nonnegative, but the signs for other Legendre coefficients are mixed. While the magnitudes of multipole amplitudes for $\ell > 1$ do tend to decrease, their coefficients in Eq. (4) tend to increase with ℓ. Thus, convergence becomes a delicate numerical issue.

Under the present conditions, we find that $\mathrm{Re}(M_{1-}E_{1+}^*)$ is the most important contribution to $\tilde{R}_{EM}$ neglected by M_{1+} dominance and is approximately -40% of the leading term. Thus, M_{1+} dominance is not very accurate. The fact that f_{EM} approaches 0.88 for $\epsilon = 0$ is actually nothing more than a lucky conspiracy among the magnitudes and signs for a very large number of smaller terms, many of which are not especially small individually. However, most experiments omit Rosenbluth separation. Similarly, the second most important contribution to $\tilde{R}_{SM}$ is $\mathrm{Re}(S_{0+}E_{2-}^*)$ but is only about 6% of the leading term; hence, f_{SM} converges more rapidly. The details of this analysis are obviously model dependent, but qualitatively similar results are obtained for other models as well. Although f_{EM} for $\epsilon = 0$ is slightly closer to unity than f_{SM} for the present analysis, the greater susceptibility of $\tilde{R}_{EM}$ to truncation errors through its reliance upon delicate cancellations suggests that the traditional Legendre analysis is intrinsically less reliable for R_{EM} than for R_{SM}, with or without Rosenbluth separation.

It is often argued that the traditional Legendre analysis should be more accurate for the isospin-3/2 channel than for the $p\pi^0$ reaction because the resonant multipoles should share a common phase and become pure imaginary at the physical mass, thereby suppressing background contributions. Leaving aside the propagation of errors involved in extracting isospin-3/2 amplitudes by combining two independent experiments, we can address the intrinsic accuracy of this analysis method using model calculations also. The convergence of the accuracy parameters for isospin-3/2 quadrupole ratios is examined in Table III. Again we find that Rosenbluth separation is required for $\tilde{R}_{EM}$. Interestingly, f_{EM} deteriorates as $\ell_{\max}$ increases and the final accuracy of $\tilde{R}_{EM}$ is worse for isospin-3/2 than for $p\pi^0$ even with $\epsilon = 0$. The cancellations are severe, the method is unstable, and calculations for $\tilde{R}_{EM}$ are highly model-dependent.

TABLE III: Convergence of quadrupole estimators for isospin-3/2. Multipole amplitudes from MAID2003 were used for $W = 1.232$ GeV and $Q^2 = 1.0$ (GeV/c)^2. Legendre coefficients are in units of $(\mu b)^{1/2}$.

	$\epsilon = 0.95$		$\epsilon = 0$	
$\ell_{\max}$	f_{EM}	f_{SM}	f_{EM}	f_{SM}
1	0.717	0.971	0.993	1.007
2	0.516	0.868	0.831	0.900
3	0.484	0.881	0.801	0.914
4	0.452	0.871	0.767	0.903
5	0.447	0.872	0.763	0.905

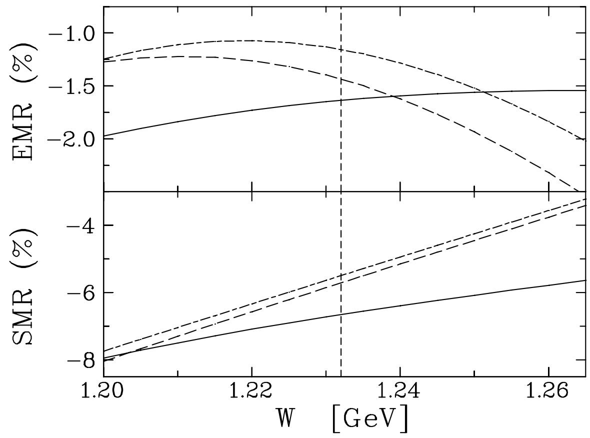

FIG. 1: W dependence of quadrupole estimators for $p\pi^0$ at $Q^2 = 1.0$ (GeV/c)^2 using MAID2003 multipoles. Solid curves show R_{EM} and R_{SM} while dashed and dash-dotted curves show $\tilde{R}_{EM}$ and $\tilde{R}_{SM}$ for $\ell \leq 5$ using $\epsilon = 0$ and $\epsilon = 0.9$, respectively. The vertical dashed line denotes $W = 1.232$ GeV.

Figure 1 compares traditional quadrupole estimators with $R_{EM}^{(p\pi^0)}$ and $R_{SM}^{(p\pi^0)}$ for MAID2003 at $Q^2 = 1.0$ (GeV/c)^2. Ideally the estimators would be most accurate in the immediate vicinity of the physical mass, $W = M_\Delta \approx 1.232$ GeV, but neither actually has that property. Rosenbluth separation is most important for R_{EM}, but even with separation the residual error at M_Δ is significant at the level of experimental precision that is now possible.

Similarly, Fig. 2 shows the Q^2 dependence for the accuracies of the traditional quadrupole estimators using MAID2003 $p\pi^0$ multipole amplitudes for $W = 1.232$ GeV. Solid curves use $\epsilon = 0$, corresponding to Rosenbluth separation, while dashed curves use $\epsilon = 0.9$, typical of many experiments. Even with Rosenbluth separation, neither quadrupole estimator can be trusted to better than about 20% and their accuracy deteriorates at larger Q^2 as M_{1+} dominance breaks down. Therefore, truncation errors can seriously affect the Q^2 dependence of quadrupole amplitudes deduced from Legendre coefficients. The most extensive recent study of $p\pi^0$ quadrupole ratios for $0.4 \leq Q^2 \leq 1.8$ (GeV/c)^2 used Eq. (6) without Rosenbluth separation [2]. We do not advocate adjustment of such results using Fig. 2, at least at this time, because the shapes of f_{EM} and f_{SM} are model dependent and MAID2003 does not describe all of the low-lying multipole amplitudes at $Q^2 = 1$ (GeV/c)^2 from Ref. [7] sufficiently well to be confident of its predictions for the Q^2 dependencies of these ratios. Instead, we claim that accurate measurements of the quadrupole ratios require multipole analysis of both polarization and cross section data.

Finally, other simple estimators

$$\mathrm{Re}E_{0+}M_{1+}^* \approx A_1^T/2 \tag{8a}$$

$$\mathrm{Re}S_{0+}M_{1+}^* \approx A_0^{LT} \tag{8b}$$

$$\mathrm{Re}M_{1-}M_{1+}^* \approx -(2A_0^T + 2A_0^{TT} + A_2^T)/8 \tag{8c}$$

based upon M_{1+} dominance and *sp* truncation are sometimes quoted [2, 11]. Note that Rosenbluth separation is required. However, using MAID2003 $p\pi^0$ multipoles at $(W, Q^2) = (1.232, 1.0)$ with $\ell \leq 5$, the ratios between the right- and left-hand sides of Eq. (8) are 1.74, −0.77, and 9.75. Most notably, the numerical contribution of $\mathrm{Re}M_{1-}M_{1+}^*$ is only the fifth largest term in the multipole expansion of the specified combination of Legendre coefficients. Therefore, these estimators are worthless under these conditions.

In summary, we have performed a detailed numerical analysis of truncation errors in quadrupole ratios deduced

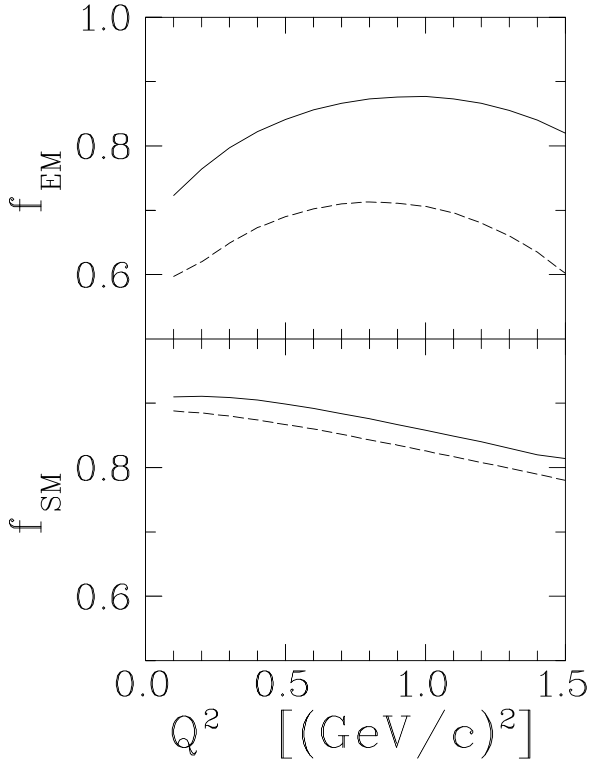

FIG. 2: Accuracy of traditional quadrupole estimators for $p\pi^0$ at $W = 1.232$ GeV using MAID2003 multipoles. Solid curves use $\epsilon = 0$ and dashed curves use $\epsilon = 0.9$.

from Legendre coefficients fit to cross section angular distributions. We find that neither M_{1+} dominance nor sp truncation is reliable and that one cannot expect better than 20% accuracy from this method. Truncation errors are especially important for R_{EM}. Furthermore, accurate results for R_{EM} also require Rosenbluth separation, which was not performed in recent studies of the Q^2 dependence of the quadrupole ratios. The accuracy of truncated Legendre analyses of E_{0+}, S_{0+}, and especially M_{1-} is even worse. Polarization data for pion electroproduction are needed to perform nearly model-independent multipole analyses that provide complex amplitudes and do not depend upon unjustifiable truncation schemes.

Acknowledgments

The support of the U.S. National Science Foundation under grant PHY-0140010 is gratefully acknowledged.

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