Map Methods for Accelerators

Alex Dragt
University of Maryland

Abstract

An overview is given of accelerator design with special emphasis on the Large Hadron Collider, the world's largest accelerator scheduled to become operational in 2007, and on the proposed International Linear Collider.

An overview is also given of how map, Lie algebraic, and symplectic integration methods are used in accelerator design.

Introduction

Particle accelerators have broad applications including manufacturing, biology, medicine, chemistry, material science, nuclear physics, and high-energy physics. They range in size from small (almost table-top) cyclotrons to the Large Hadron Collider (LHC) under construction at CERN and the proposed International Linear Collider (ILC). Worldwide, accelerators number in the several hundreds. Synchrotron light sources and free-electron lasers are particularly common.

Accelerators use magnetic fields to guide and confine a charged-particle beam, and the electric fields in radio frequency (RF) cavities to accelerate the beam.

Magnets include dipoles, quadrupoles, solenoids, sextupoles, octupoles, etc. They can be normal or superconducting (1.9 degrees Kelvin).

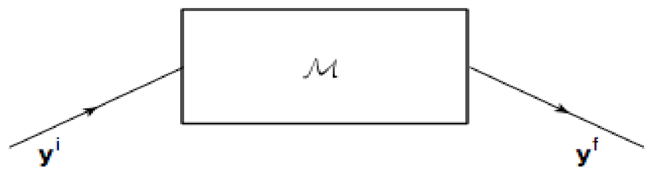
RF cavities can also be normal or superconducting.

The various beam-line elements (magets, intervening drift spaces, and RF cavities) are arranged sequentially to form what is called a lattice.

Accelerators come in two basic geometries: linear (single-pass) machines called Linacs, and circular (multi-pass) machines.

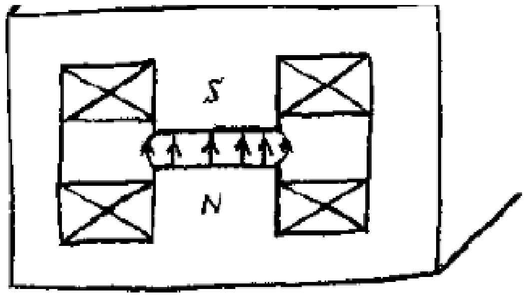
To achieve very high energies for electrons or positrons, only linear machines can be used. High-energy electrons/positrons in a circular machine would radiate energy, due to synchrotron radiation, faster than it could be re-supplied by RF cavities. (Circular motion implies continuous inward acceleration, and accelerated particles radiate.)

For heavier particles (muons, protons, ions) synchrotron radiation energy loss is much less, and the use of circular machines is advantageous. In a circular machine particles can be made to pass through the same RF cavity multiple times thereby gaining energy on each pass.

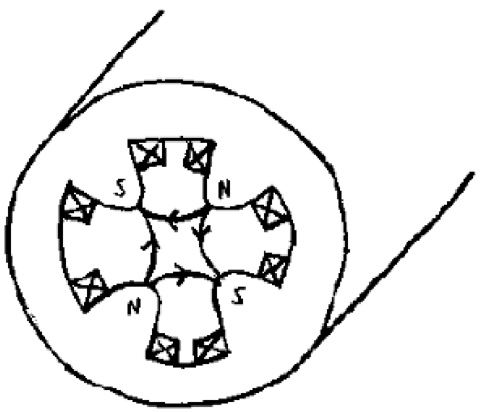


- The action of each beam-line element can be described by a (symplectic) map M. Charged particle motion is Hamiltonian, and Hamiltonian flows generate symplectic maps.

Elements that Produce Linear Maps

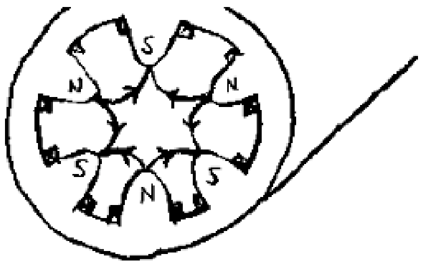


- DIPOLE MAGNET
- Provides bending

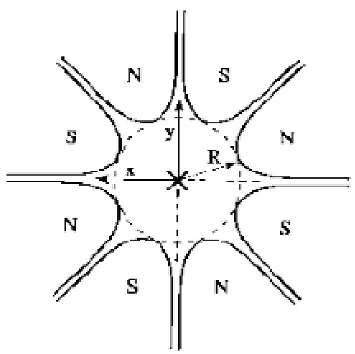


- QUADRUPOLE MAGNET
- Provides focus and defocus

Elements that Produce Nonlinear Maps

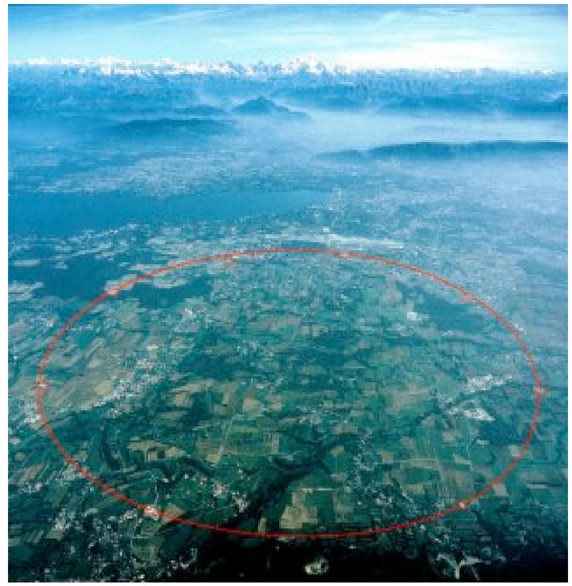


- SEXTUPOLE MAGNET
- Provides quadratic nonlinearity/correction

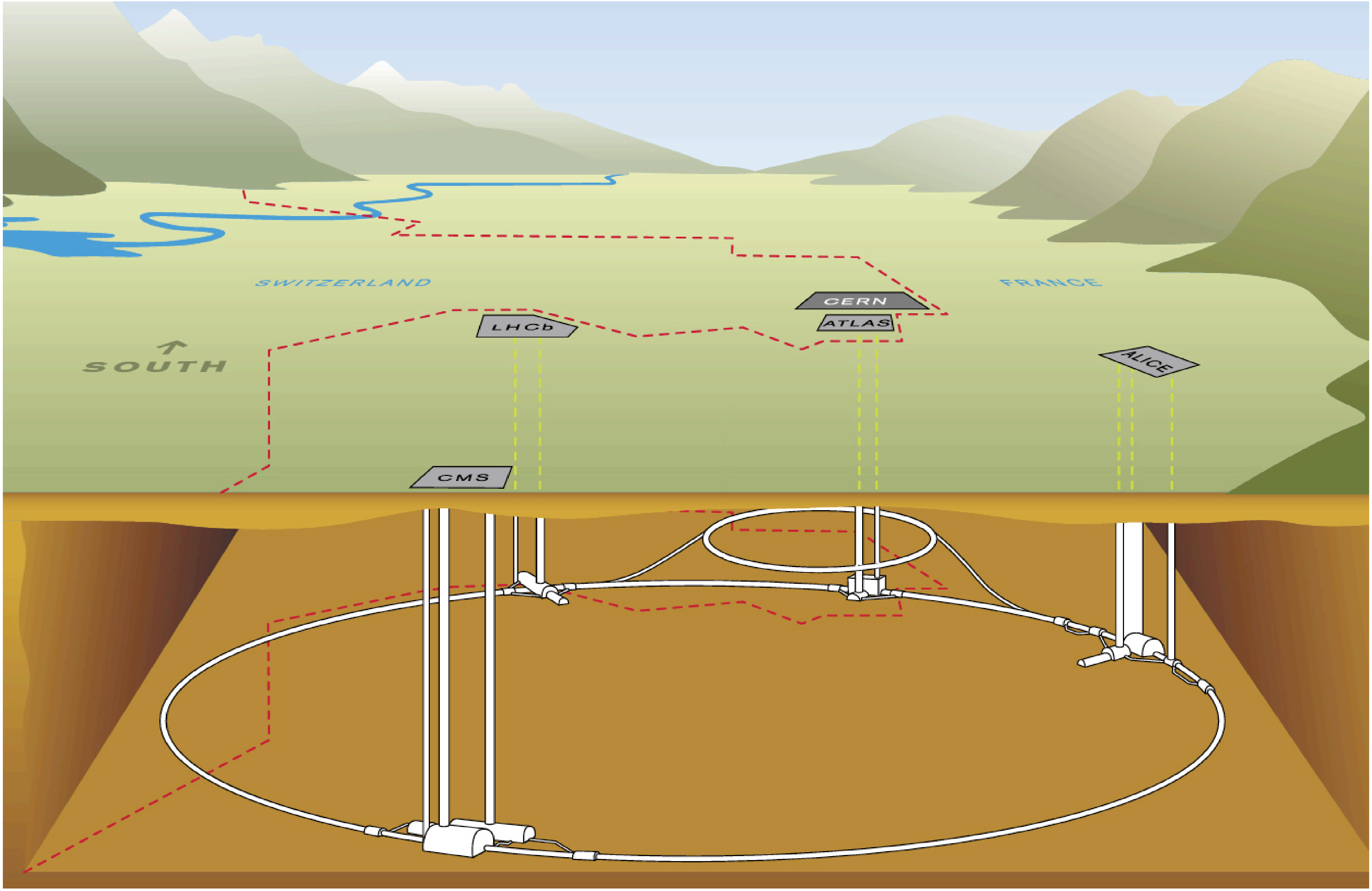


- OCTUPOLE MAGNET
- Provides cubic nonlinearity/correction

Overview of the LHC at CERN

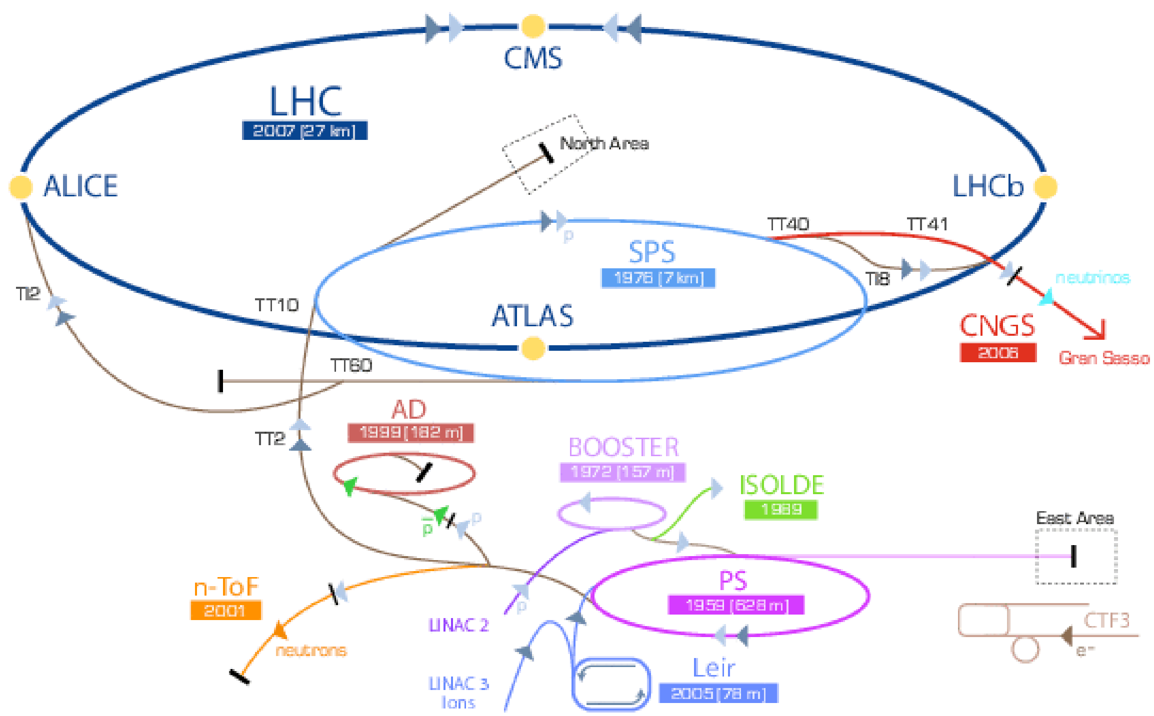


- The LHC is 27 km in circumference. It consists of two rings. One ring contains protons circulating clockwise, and the other contains protons circulating counterclockwise. They collide, within various detectors, with an energy of 14 TeV.



- Underview of the LHC at CERN

CERN Accelerator Complex



p (proton) ion neutrons p̄ (antiproton) proton/antiproton conversion neutrinos electron

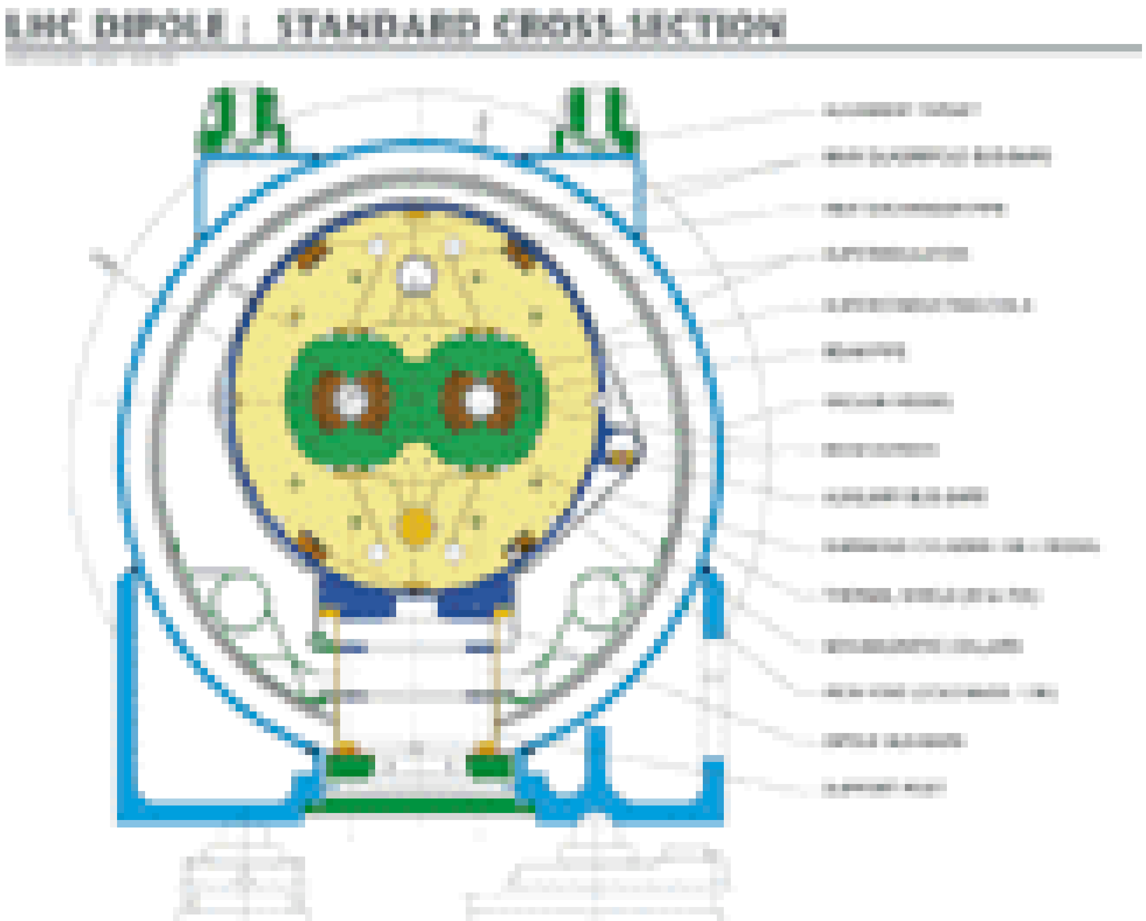
LHC Large Hadron Collider SPS Super Proton Synchrotron PS Proton Synchrotron

AD Antiproton Decelerator CTF3 Clic Test Facility CNGS Cern Neutrinos to Gran Sasso ISOLDE Isotope Separator OnLine DEvice

LEIR Low Energy Ion Ring LINAC LINear ACcelerator n-ToF Neutrons Time Of Flight

LHC DIPOLE testing at CERN

Drawing of dipole showing two beam pipes in a common cryo-module cooled to 1.9 degree K

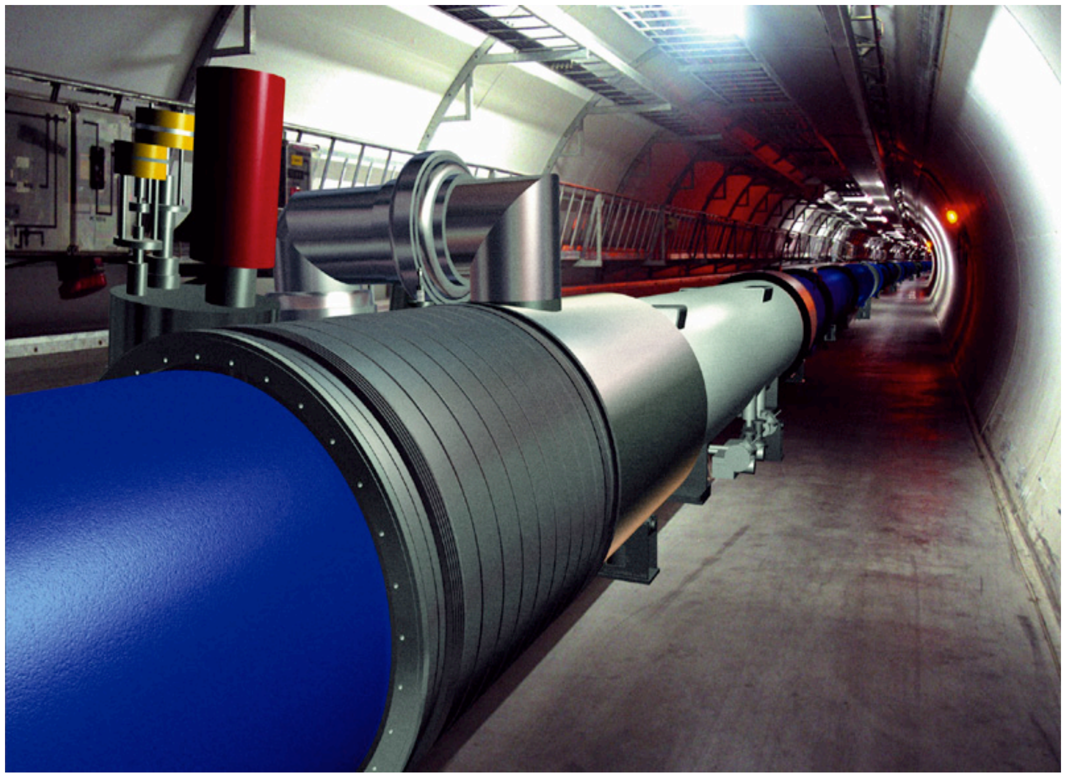


Japanese/American LHC Quad

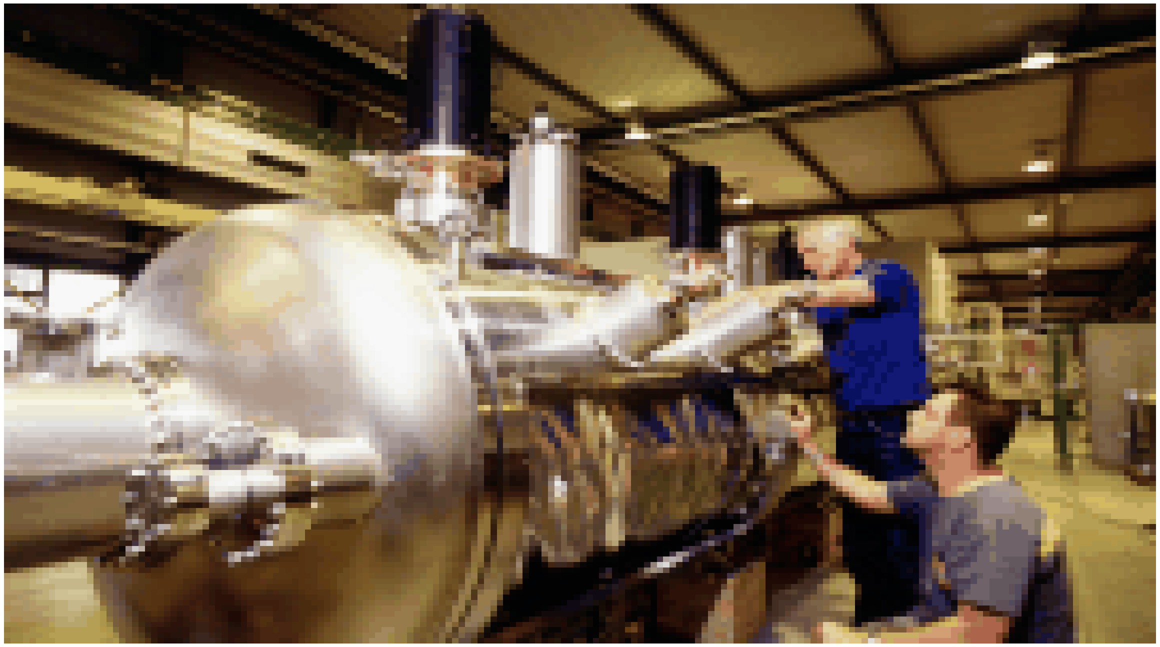
Dipole being lowered into LHC tunnel

- The LHC has over 4500 magnets in each ring. The main dipoles are 15 meters long and have a field strength of 8.33 T.

View of lattice as installed in LHC tunnel

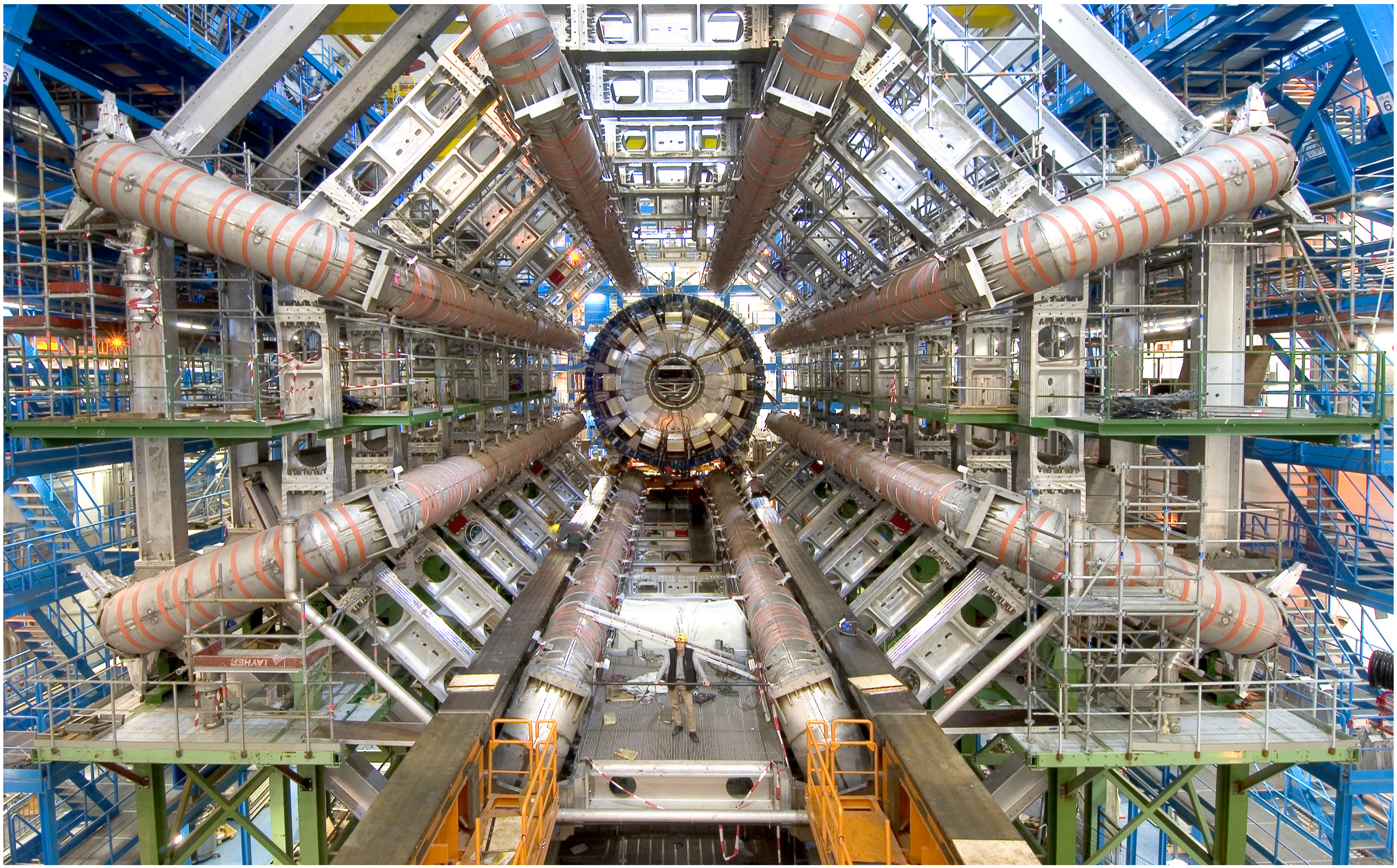


Superconducting RF Cavites



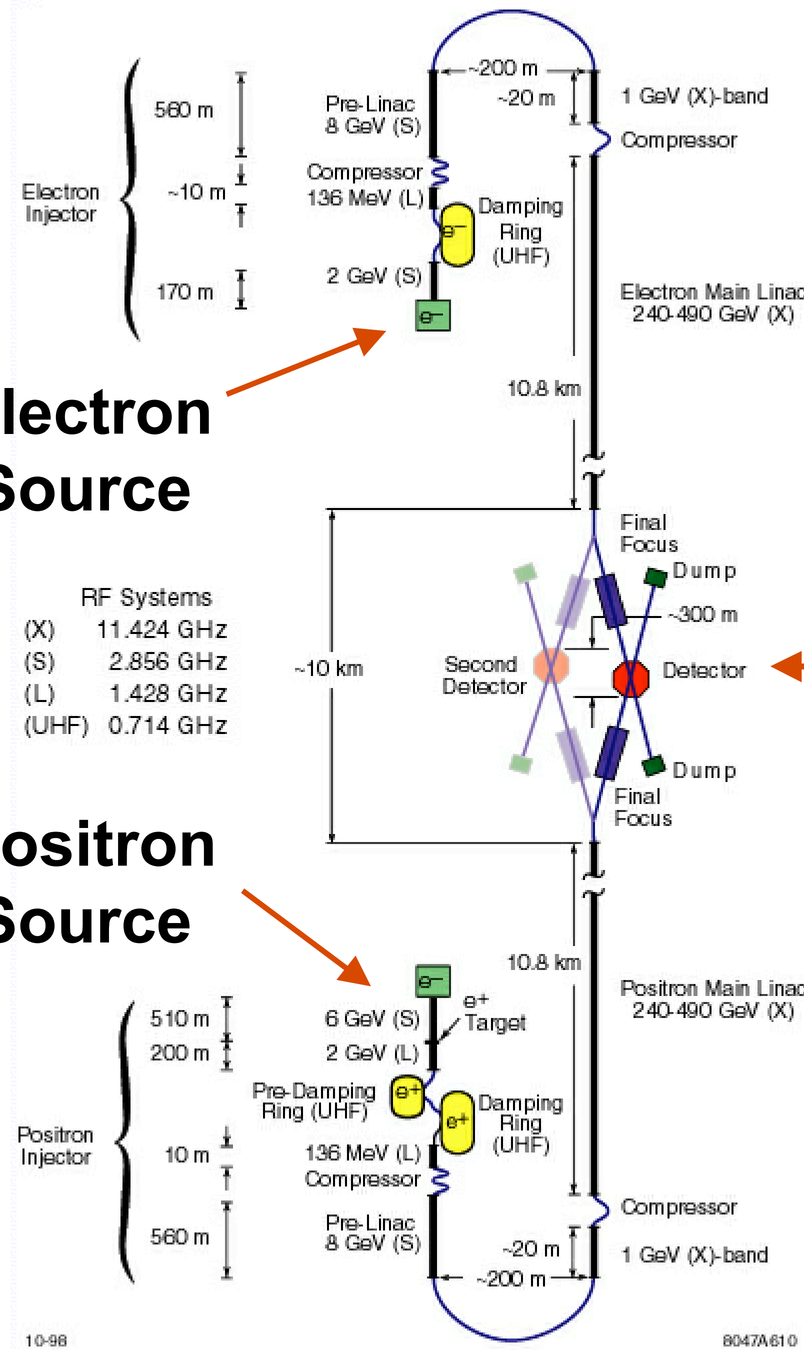
Four cavities form a module

View of Atlas Detector Construction



Next Linear Collider Schematic Diagram

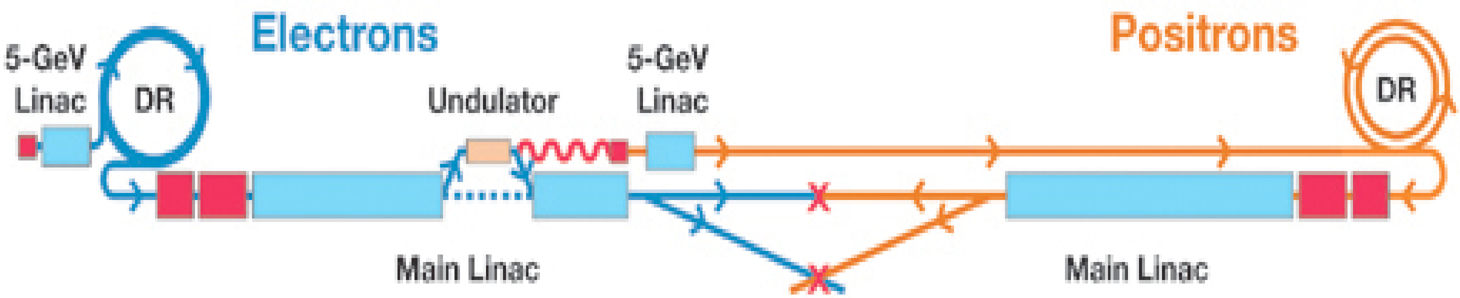
Electron Source



Positron Source

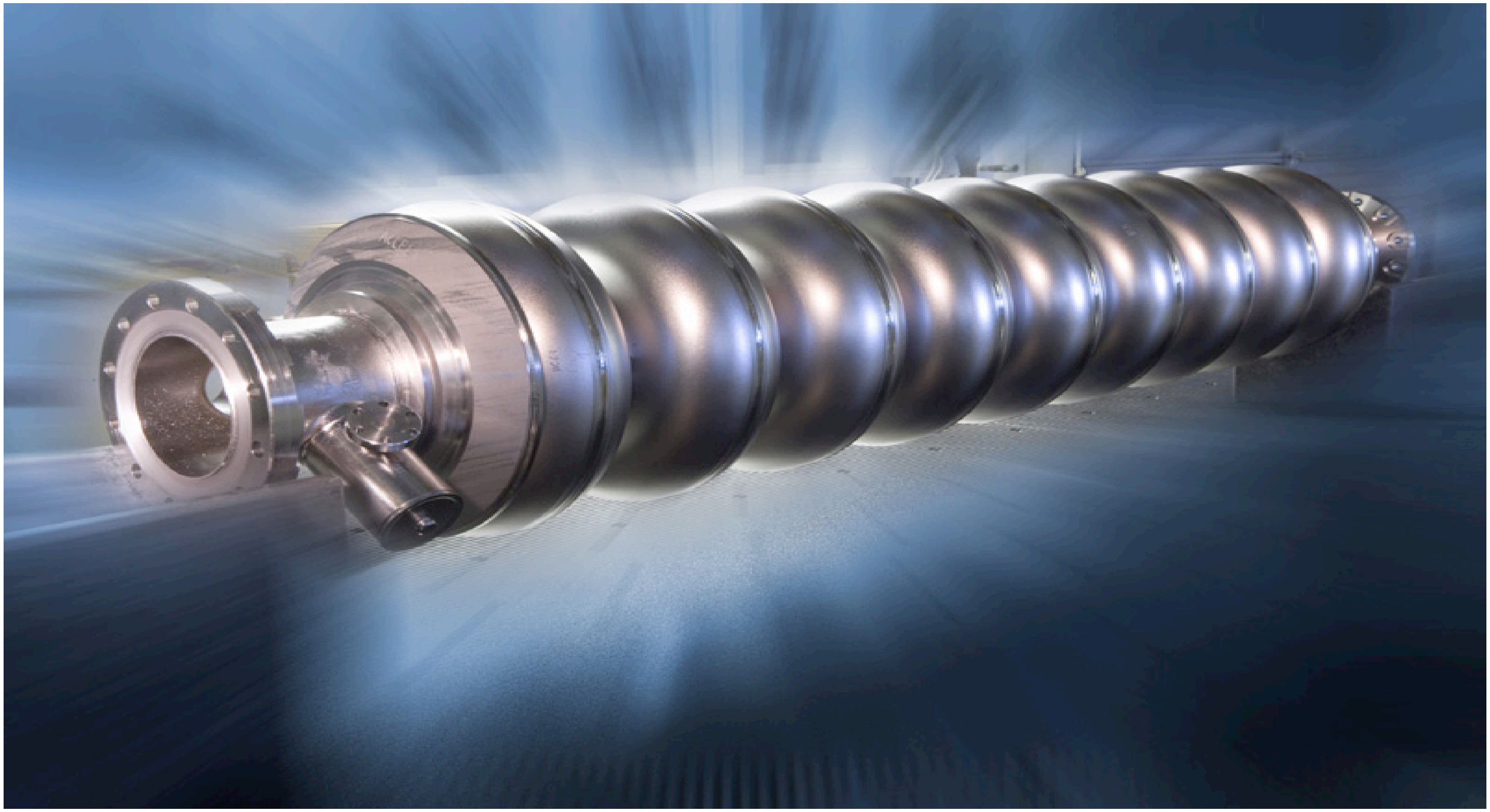
Matter Anti-Matter Interaction Point @ 0.5 TeV

InterLab 99
Stanford Linear Accelerator Center



Electrons
Positrons
5-GeV
Linac
DR
Undulator
5-GeV
Linac
DR
Main Linac
Main Linac


© 2005 Shigemi Numazawa



Fitting Wiggler Data

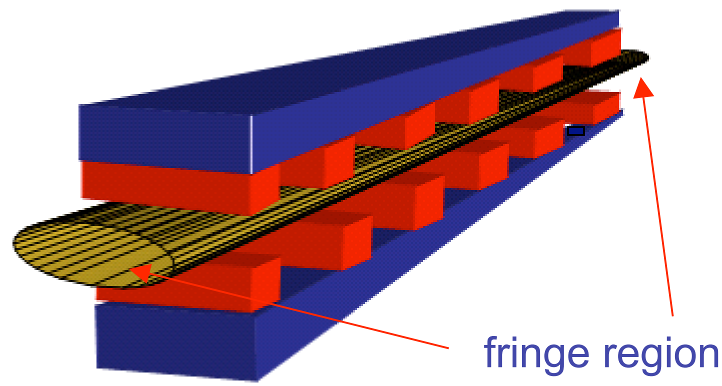
- Data on regular Cartesian grid

4.8cm in x, dx=0.4cm

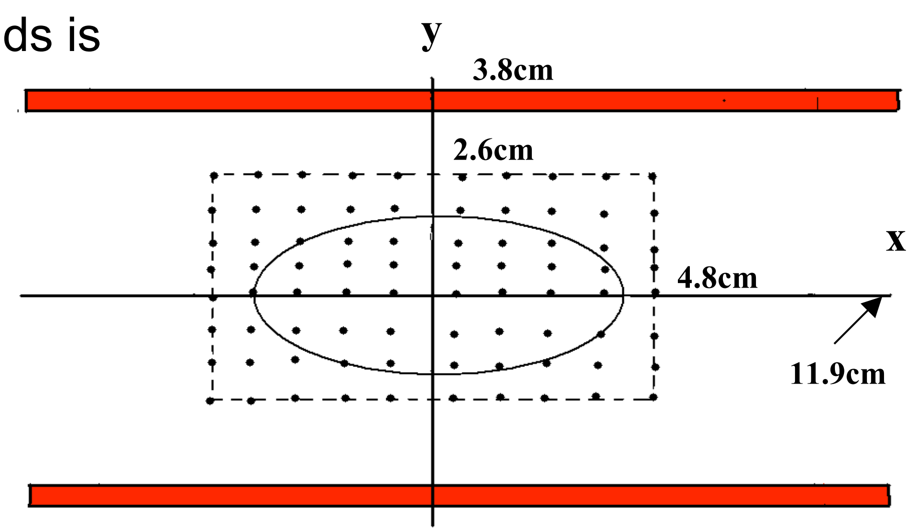
2.6cm in y, dy=0.2cm

480cm in z, dz=0.2cm

- Field components Bx, By, Bz in one quadrant given to a precision of 0.05G.



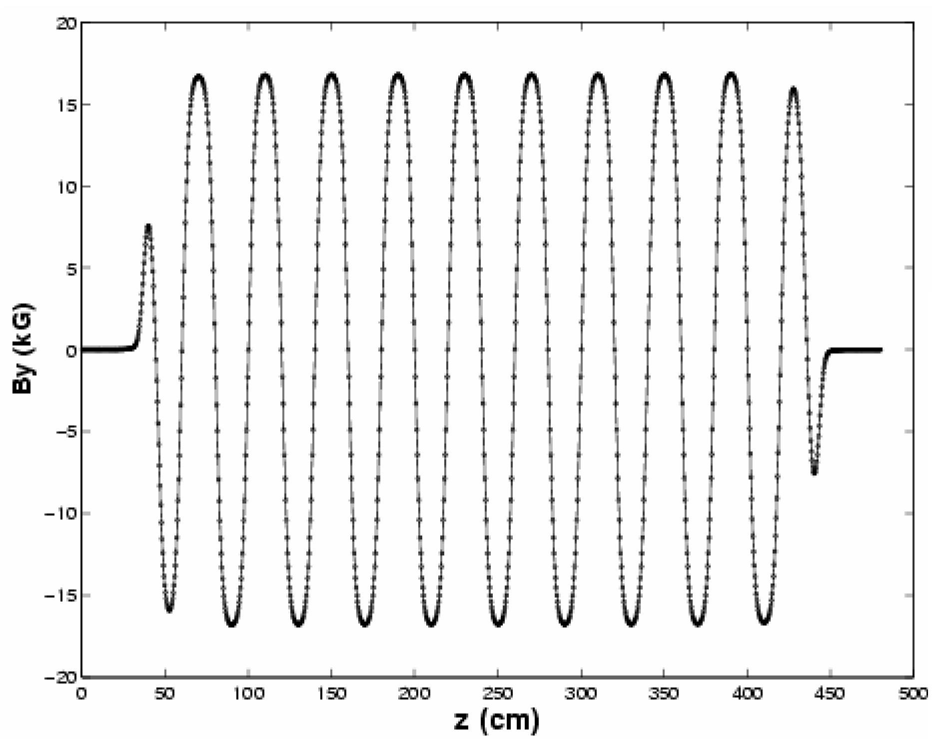
- Place an imaginary elliptic cylinder between pole faces, extending beyond the ends of the magnet far enough that the field at the ends is effectively zero.



- Fit data onto elliptic cylindrical surface using bicubic interpolation to obtain the normal component on the surface.

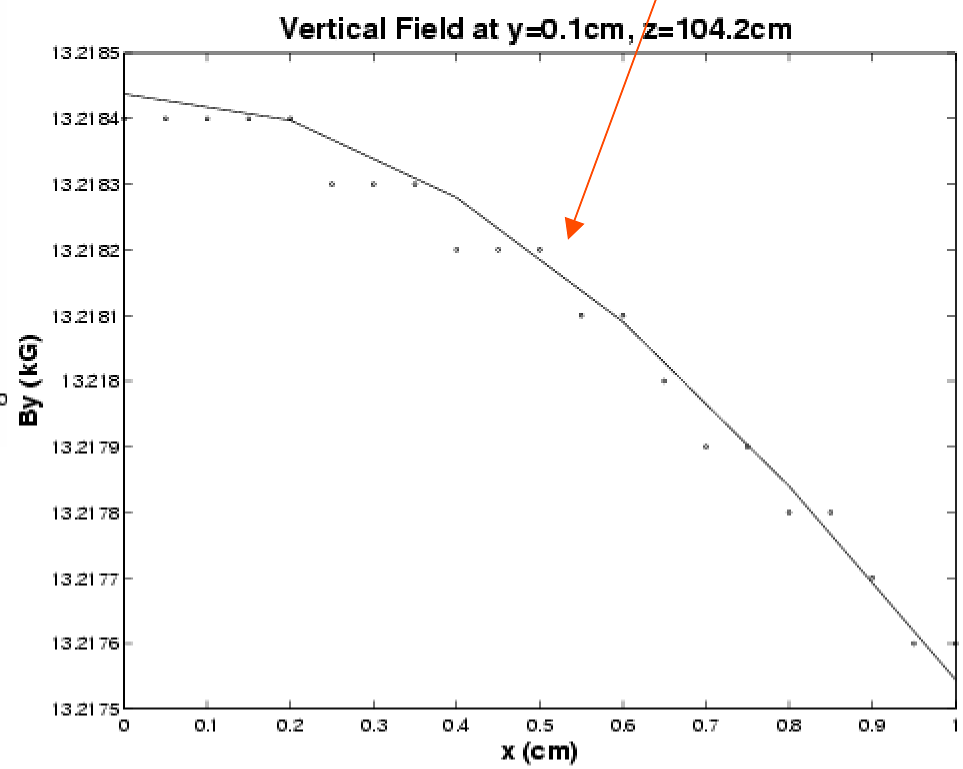
- Compute the interior vector potential and all its desired derivatives from surface data.

Fit to the Proposed ILC Wiggler Field Using Elliptical Cylinder



Fit to vertical field By
at x=0.4cm, y=0.2cm.

Note precision of data.



Mathematical Tools

Write $z = \{q, p\}$. Let f be any function of z, and suppose g is any other function. Define a Lie operator (Hamiltonian vector field) $:f:$ by the rule $:f:g = [f, g]$ where [,] denotes the Poisson bracket.

Define $\exp{:f:}$, called a Lie transformation, by the series

$$\exp{:f:} := \sum_{0}^{\infty} {:f:}^n / n!$$

Define a map M that sends z to $\bar{z}$ by writing

$$\bar{z} = Mz = (\exp{:f:})z.$$

Theorem: M is a symplectic map for any f.

Theorem: Any symplectic M map can be written as a product of Lie transformations,

$$M = (\exp : f_1 :)(\exp : f_2 :)(\exp : f_3 :)(\exp : f_4 :)..., \quad (*)$$

where each f_n is a homogeneous polynomial of degree n.

The polynomial f_1 generates translations, f_2 generates linear transformations, f_3 generates quadratic and higher-order terms, f_4 generates cubic and higher-order terms, etc. Thus, the effect of any beam-line element can be described by a collection of homogeneous polynomials.

Theorem: Suppose H is the Hamiltonian whose flow generates $\mathcal{M}$. Then $\mathcal{M}$ has the equation of motion

$$\dot{\mathcal{M}} = \mathcal{M} :-H: \qquad (**)$$

Together (*) and (**) give equations of motion for the f_n. They are known explicitly through $n = 8$, and can be integrated analytically if H is simple, otherwise numerically.

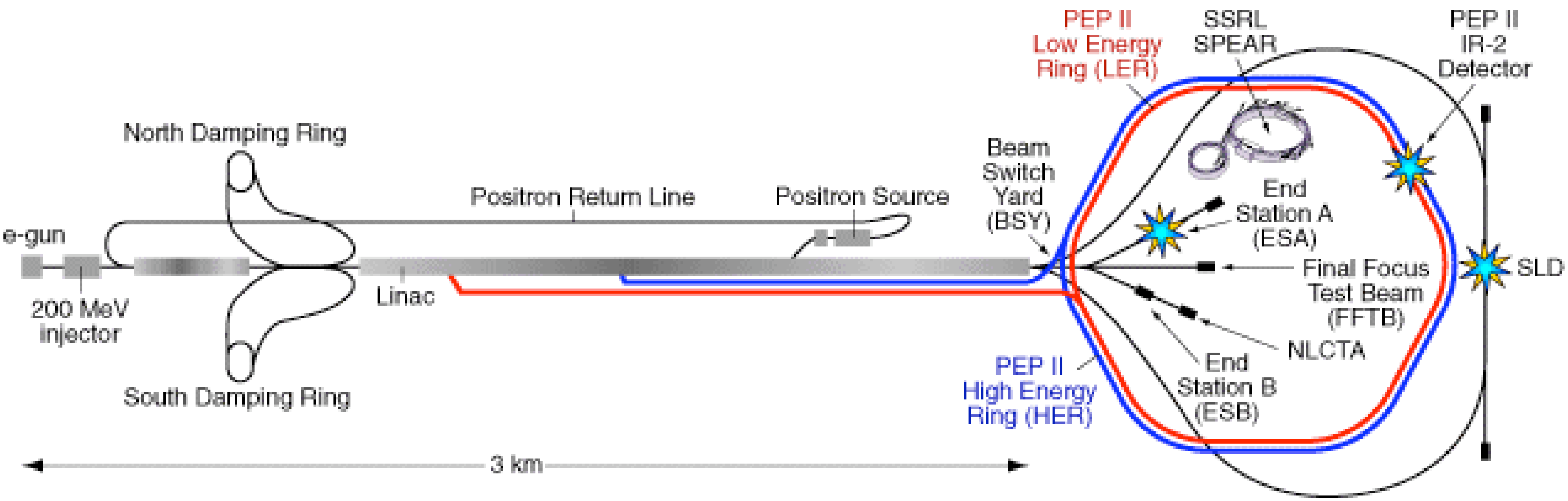
Theorem: Given any two maps $\mathcal{M}_f$ and $\mathcal{M}_g$ characterized by homogeneous polynomials f_l and g_m, there are (unique) homogeneous polynomials h_m such that

$$\mathcal{M}_h = \mathcal{M}_f \mathcal{M}_g .$$

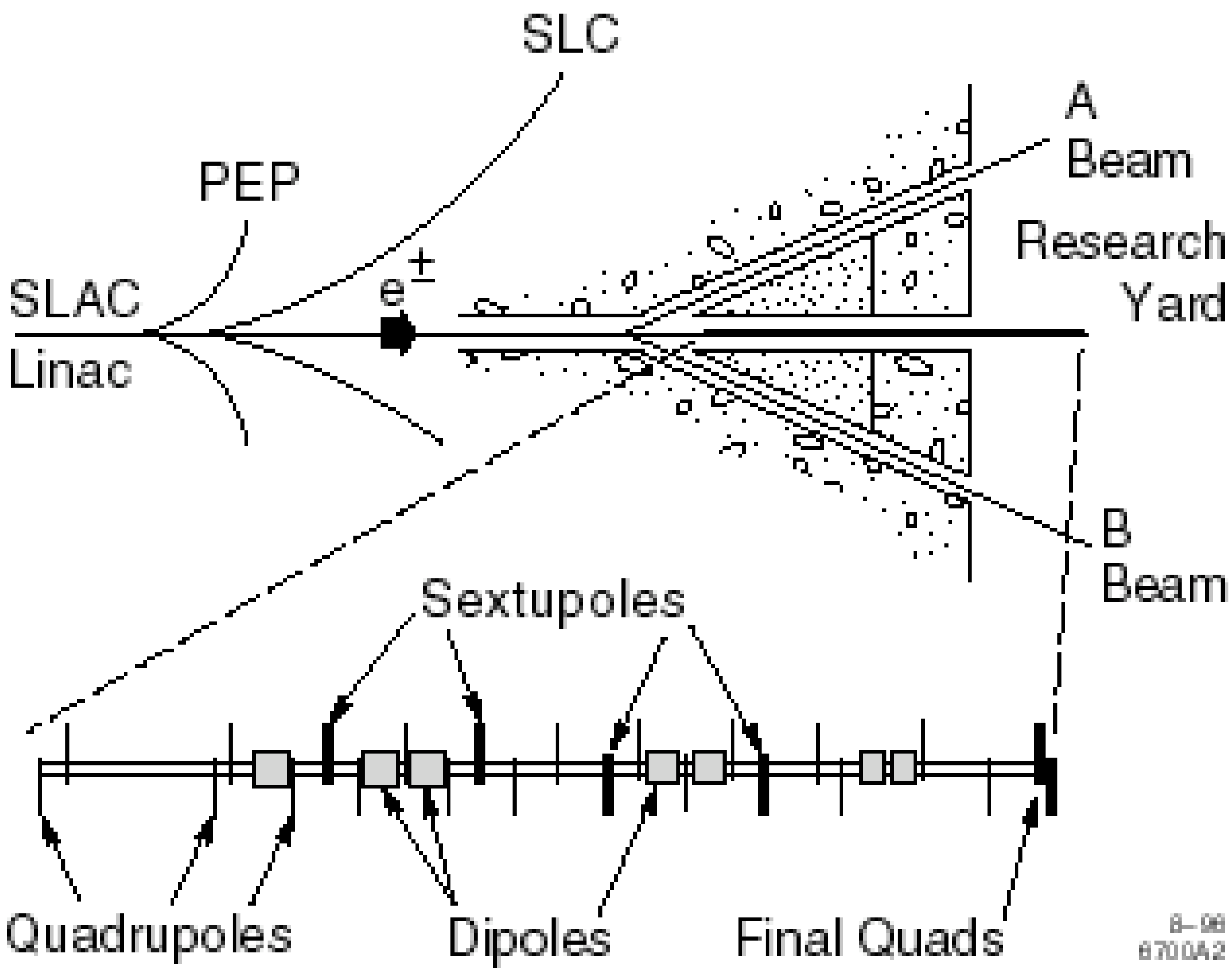
They can be found using the BCH formula or other methods. Explicit results are known through $n = 8$. Thus, maps for individual beam-line elements can be combined to find the net map that describes a section of or even an entire beamline or ring.

In the case of a linear collider (spot - forming system), the net map for the final focus system can be used to compute and optimize interaction - region spot size (10's of Angstroms). This method was used to optimize the performance of the Stanford Linear Collider, and in the preliminary design of the ILC final focus system.

In the case of a circular machine, the full one - turn map can be used to predict and optimize all desired machine properties. This method was used in the design and operation of the Stanford B factory, a circular electron - positron collider.



PEP II
Low Energy
Ring (LER)
SSRL
SPEAR
PEP II
IR-2
Detector
North Damping Ring
Beam
Switch
Yard
(BSY)
End
Station A
(ESA)
Positron Return Line
Positron Source
e-gun
Final Focus
Test Beam
(FFTB)
SLD
200 MeV
injector
Linac
NLCTA
End
Station B
(ESB)
PEP II
High Energy
Ring (HER)
South Damping Ring
3 km


SLC
PEP
SLAC
Linac
e±
A
Beam
Research
Yard
B
Beam
Sextupoles
Quadrupoles
Dipoles
Final Quads
8–96
8700A2


N

Normal form analysis

Observation: Suppose M is the one-turn map for a ring. Then there is a symplectic map A such that the map N given by

$$N = AMA^{-1}$$

has a simple form called the normal form. This is the nonlinear analog of diagonalizing a matrix.

The map N describes global properties of the ring (tunes, anharmonicities, chromaticities, phase - slip factors). The map A describes local properties (linear and nonlinear lattice functions). For example, will the beam hit the wall of the beam pipe?

Long - term behavior of storage - ring orbits

The LHC is expected to circulate beam for about 8 hours between refillings. During this time particles make more than 10^{10} betatron oscillations. Can this assumed stability be assured in advance by theoretical and numerical calculations? Doing so is like predicting the stability of the solar system since the Big Bang.

Let M be the one - turn map for a ring.

Need to compute $M^n z$ for large n and a variety of initial conditions z.

Need to preserve the symplectic condition, which rules out using directly a finite Taylor expansion for M which, unfortunately, is all we really have.

Methods for symplectifying a truncated Taylor map

Construct a mixed - variable polynomial generating function whose results,when made explicit, agree with the given Taylor series to the order which it is known. Find its results to all orders numerically using Newton's method. Use of this method for LHC orbits has not been explored. It is about to be applied to realistic one - turn maps for the ILC damping rings including all nonlinear wiggler effects associated with real (not idealized) magnetic fields.

Make a Cremona map approximation. Cremona maps are maps that are polynomial and also also exactly symplectic. Since they are polynomial, they can be evaluated rapidly and exactly. Since they are symplectic, they preserve the underlying Hamiltonian structure.

Construction: Given a truncated Taylor approximation to a symplectic map, there is a Cremona map whose lower-order terms agree with the truncated Taylor map to the order to which it is known, and whose higher-order terms are small in size and finite in number (remember, a Cremona map is polynomial).

Its construction requires group theory [SU(2) and SU(3)] and cubature formulas for the two sphere and the manifold SU(3)/O(3). (Cubature formulas are quadrature formulas generalized to more than one dimension).

There is still much to be learned about this method.
It has been applied sucessfully to LHC orbits.
Its results agree with crude element - by - element "kick - map" results when the equivalent crude one - turn map is used, but is 10 times faster.
Moreover, it can be applied with equal efficiency to realistic one - turn maps that include all fringe - field and error effects.

Symplectic Integration

Symplectic integrators are well known for Hamiltonians of the form $T(p)+V(q)$. The usual Hamiltonian for charged-particle motion in an electromagnetic field is not of this form. It involves square root terms that contain both p and q.

However, if the time is replaced by a new independent variable τ, there is a Hamiltonian that treats all 4 space - time components x^μ and their conjugate momenta p_μ as dependent variables. It has the form

$$H_R = [1/(2mc)][p_\mu - qA_\mu(x)][p_\nu - qA_\nu(x)]g^{\mu\nu}.$$

Since H_R is τ independent, it must be a constant of motion. Orbits of physical interest are those that satisfy $H_R = -mc/2$.

The mechanical 4 - momentum is given by

$$p_\mu^{mech} = p_\mu - qA_\mu .$$

H_R must have the value $(-mc/2)$ in order for the particle to have the correct mass:

$$(p_\mu^{mech})(p_\nu^{mech})g^{\mu\nu} = -(mc)^2 .$$

Forest and co-authors have found a symplectic integrator S_2 for H_R that is symmetric and locally correct through terms of order h^2. Since S_2 is symmetric, Yoshida's method can be applied to it to construct higher-order symmetric symplectic integrators S_4, etc.

S_2 is a product of 21 symplectic factors that are either drifts or gauge transformations. Each factor depends only on x or p, and therefore its effect can be evaluated exactly.

S_2, or its Yoshida extensions, can be used to integrate particle orbits provided the step size h is small enough to ensure that H_R remains close to the required physical value $(-mc/2)$.

While this integration is taking place, auxiliary code due to Forest can be used to compute the Taylor map about the orbit. This can be done to any desired order provided the derivatives of the $A_\mu(x)$ about the orbit are available to the desired order. The Taylor map can in turn be converted into a factored product Lie map. Thus, the work of Forest provides bridges between the various approaches.