

# Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics

---

Alex J. Dragt

*University of Maryland, College Park*

[www.physics.umd.edu/dsat/](http://www.physics.umd.edu/dsat/)



6 November 2009

Alex J. Dragt  
Dynamical Systems and Accelerator Theory Group  
Department of Physics  
University of Maryland  
College Park, Maryland 20742

[www.physics.umd.edu/dsat](http://www.physics.umd.edu/dsat)

Supported in part by U. S. Department of Energy Grant DE-FG02-96ER40949.

---

---

© 1991 by Alex J. Dragt.

All rights reserved.

# Contents

<i>Preface</i>	xlix
<b>1 Introductory Concepts</b>	<b>1</b>
1.1 Transfer Maps . . . . .	1
1.1.1 Maps and Dynamics . . . . .	1
1.1.2 Maps and Accelerator Physics . . . . .	4
1.1.3 Maps and Geometry . . . . .	5
1.2 Map Iteration and Other Background Material . . . . .	6
1.2.1 Logistic Map . . . . .	6
1.2.2 Complex Logistic Map . . . . .	14
1.2.3 Simplest Nonlinear Symplectic Map . . . . .	19
1.2.4 Goal for Use of Maps in Accelerator Physics . . . . .	23
1.2.5 Maps from Hamiltonian Differential Equations . . . . .	24
1.3 Essential Theorems for Differential Equations . . . . .	36
1.4 Transfer Maps Produced by Differential Equations . . . . .	41
1.4.1 Map for Simple Harmonic Oscillator . . . . .	42
1.4.2 Maps for Monomial Hamiltonians . . . . .	43
1.4.3 Stroboscopic Maps and Duffing Equation Example . . . . .	44
1.5 Hamilton's equations with time as an independent variable . . . . .	84
1.6 Hamilton's equations with a coordinate as an independent variable . . . . .	93
1.7 Definition of Poisson Bracket . . . . .	110
<b>2 Numerical Integration</b>	<b>129</b>
2.1 The General Problem . . . . .	130
2.2 A Crude Solution Due to Euler . . . . .	130
2.3 Runge-Kutta Methods . . . . .	137
2.4 Finite-Difference Methods . . . . .	144
2.4.1 Finite-Difference Procedure . . . . .	144
2.4.2 Derivation and Error Analysis . . . . .	151
2.5 (Automatic) Choice and Change of Step Size and Order . . . . .	159
2.5.1 Change of Step Size in Runge-Kutta . . . . .	160
2.5.2 Adaptive Finite Difference Methods . . . . .	160
2.5.3 Jet Formulation . . . . .	161
2.5.4 Virtues of Jet Formulation . . . . .	167
2.5.5 Advice to the Novice . . . . .	170

2.6	Extrapolation Methods . . . . .	172
2.6.1	Overview . . . . .	172
2.6.2	Making a Meso Step . . . . .	173
2.6.3	Summary . . . . .	176
2.6.4	Again, Advice to the Novice . . . . .	176
2.7	Things Not Covered . . . . .	177
<b>3</b>	<b>Symplectic Matrices</b>	<b>187</b>
3.1	Definitions . . . . .	187
3.2	Variants . . . . .	189
3.3	Simple Symplectic Restrictions and Symplectic Factorization . . . . .	192
3.3.1	Large-Block Formulation . . . . .	192
3.3.2	Symplectic Block Factorization . . . . .	193
3.3.3	Symplectic Matrices Have Determinant +1 . . . . .	195
3.3.4	Small-Block Formulation . . . . .	196
3.4	Eigenvalue Spectrum . . . . .	197
3.4.1	Background . . . . .	197
3.4.2	The $2 \times 2$ Case . . . . .	198
3.4.3	The $4 \times 4$ and $2n \times 2n$ Cases . . . . .	199
3.4.4	Further Symplectic Restrictions . . . . .	203
3.5	Eigenvector Structure and Normal Forms . . . . .	208
3.5.1	Eigenvector Basis . . . . .	208
3.5.2	$J$ -Based Angular Inner Product . . . . .	208
3.5.3	Use of Angular Inner Product . . . . .	209
3.5.4	Definition and Use of Signature . . . . .	210
3.5.5	Definition of Phase Advances and Tunes . . . . .	212
3.5.6	The Krein-Moser Theorem and Krein Collisions . . . . .	212
3.5.7	Normal Forms . . . . .	214
3.6	Group Properties, Dyad and Gram Matrices, and Bases . . . . .	218
3.6.1	Group Properties . . . . .	218
3.6.2	Dyad and Gram Matrices, Bases and Reciprocal Bases . . . . .	219
3.6.3	Orthonormal and Symplectic Bases . . . . .	222
3.6.4	Construction of Orthonormal Bases . . . . .	225
3.6.5	Construction of Symplectic Bases . . . . .	229
3.7	Lie Algebraic Properties . . . . .	234
3.7.1	Matrix Exponential and Logarithm . . . . .	234
3.7.2	Application to Symplectic Matrices . . . . .	236
3.7.3	Matrix Lie Algebra and Lie Group . . . . .	238
3.7.4	Abstract Definition of a Lie Algebra . . . . .	240
3.7.5	Abstract Definition of a Lie Group . . . . .	241
3.7.6	Classification of Lie Algebras . . . . .	241
3.7.7	Adjoint Representation of a Lie Algebra . . . . .	244
3.8	Exponential Representation of Symplectic Matrices . . . . .	260
3.9	Unitary Subgroup Structure . . . . .	271
3.10	Other Subgroup Structure . . . . .	278

3.11	Cayley Representation of Symplectic Matrices . . . . .	281
3.12	General Symplectic Forms, Darboux Transformations, and Variant Symplectic Groups . . . . .	283
3.12.1	General Symplectic Forms . . . . .	283
3.12.2	Darboux Transformations . . . . .	287
3.12.3	Variant Symplectic Groups . . . . .	288
<b>4</b>	<b>Matrix Exponentiation and Symplectification</b>	<b>297</b>
4.1	Exponentiation by Scaling and Squaring . . . . .	298
4.2	(Orthogonal) Polar Decomposition . . . . .	305
4.3	Symplectic Polar Decomposition . . . . .	309
4.4	Finding the Closest Symplectic Matrix . . . . .	318
4.5	Symplectification by Iteration . . . . .	333
4.5.1	Properties of Symplectification using Symplectic Polar Decomposition . . . . .	333
4.5.2	Iteration . . . . .	334
4.6	Modified Darboux Symplectification . . . . .	339
4.7	Exponential and Cayley Symplectifications . . . . .	342
4.8	Transformation (Generating) Function Symplectification . . . . .	345
<b>5</b>	<b>Lie Algebraic Structure of Classical Mechanics and Other Delights</b>	<b>351</b>
5.1	Properties of the Poisson Bracket . . . . .	351
5.2	Equations, Constants, and Integrals of Motion . . . . .	353
5.3	Lie Operators . . . . .	355
5.4	Lie Transformations . . . . .	360
5.4.1	Definition and Some Properties . . . . .	360
5.4.2	Applications . . . . .	363
5.5	Realization of the $sp(2n)$ Lie Algebra . . . . .	364
5.6	Basis for $sp(2)$ . . . . .	367
5.7	Basis for $sp(4)$ . . . . .	370
5.8	Basis for $sp(6)$ . . . . .	375
5.8.1	$U(3)$ Preliminaries . . . . .	376
5.8.2	Polynomials for $u(3)$ . . . . .	377
5.8.3	Plan for the Remaining Polynomials . . . . .	378
5.8.4	Cartan Basis for $su(3)$ . . . . .	378
5.8.5	Representations of $su(3)$ . . . . .	380
5.8.6	Weight Diagrams for First Few $su(3)$ Representations . . . . .	382
5.8.7	Weight Diagram for the General $su(3)$ Representation . . . . .	383
5.8.8	Remaining Polynomials . . . . .	387
5.9	Topology of $Sp(2n, R)$ . . . . .	401
5.10	Notational Pitfalls and Quaternions . . . . .	403
5.10.1	The Lie Algebras $sp(2n, R)$ and $usp(2n)$ . . . . .	403
5.10.2	$USp(2n)$ and Quaternions . . . . .	405
5.10.3	Quaternion Matrices . . . . .	406
5.10.4	Properties of Quaternion Matrices . . . . .	407

5.10.5	Quaternion Matrices and $USp(2n)$ . . . . .	408
5.10.6	Quaternion Inner Product and Its Preservation . . . . .	409
5.10.7	Discussion . . . . .	410
5.11	Moebius Transformations . . . . .	414
5.11.1	Definition in the Context of Complex Variables . . . . .	415
5.11.2	Matrix Extension . . . . .	416
5.11.3	Invertibility Conditions . . . . .	416
5.11.4	Transitivity . . . . .	419
5.12	Symplectic Transformations and Siegel Space . . . . .	420
5.12.1	Action of $Sp(2n, C)$ on Space of Complex Symmetric Matrices . . . . .	420
5.12.2	Siegel Space and $Sp(2n, R)$ . . . . .	421
5.12.3	Group Action on Homogeneous Space . . . . .	421
5.12.4	Homogeneous Spaces and Cosets . . . . .	423
5.12.5	Group Action on Cosets Equals Group Action on Homogeneous Space . . . . .	425
5.12.6	Application of Results to Action of $Sp(2n, R)$ on Siegel Space . . . . .	425
5.12.7	Action of $Sp(2n, R)$ on the Generalized Real Axis . . . . .	427
5.12.8	Symplectic Modular Groups . . . . .	428
5.13	Moebius Transformations Relating Symplectic and Symmetric Matrices . . . . .	433
5.13.1	Overview . . . . .	433
5.13.2	The Cayley Moebius Transformation . . . . .	434
5.13.3	Two Symplectic Forms and Their Relation by a Darboux Transformation . . . . .	435
5.13.4	The Infinite Family of Darboux Transformations . . . . .	435
5.13.5	Isotropic Vectors and Lagrangian Planes . . . . .	437
5.13.6	Connection between Symplectic Matrices and Lagrangian Planes . . . . .	438
5.13.7	Connection between Symmetric Matrices and Lagrangian Planes . . . . .	439
5.13.8	Relation between Symplectic and Symmetric Matrices and the Role of Moebius Transformations . . . . .	441
5.13.9	Completion of Tasks . . . . .	444
5.14	Matrix Symplectification Revisited . . . . .	451
5.15	Uniqueness of Cayley Moebius Transformation . . . . .	454
<b>6</b>	<b>Symplectic Maps</b> . . . . .	<b>465</b>
6.1	Preliminaries and Definitions . . . . .	465
6.1.1	Gradient Maps . . . . .	466
6.1.2	Symplectic Maps . . . . .	467
6.2	Group Properties . . . . .	471
6.3	Preservation of General Poisson Brackets . . . . .	483
6.4	Relation to Hamiltonian Flows . . . . .	485
6.4.1	Hamiltonian Flows Generate Symplectic Maps . . . . .	486
6.4.2	Any Family of Symplectic Maps Is Hamiltonian Generated . . . . .	488
6.4.3	Almost All Symplectic Maps Are Hamiltonian Generated . . . . .	492

6.5	Mixed-Variable Transformation (Generating) Functions . . . . .	494
6.5.1	Transformation Functions Produce Symplectic Maps . . . . .	494
6.5.2	Finding the Generating Hamiltonian . . . . .	497
6.5.3	Finding a Transformation Function for a Map . . . . .	499
6.6	Transformation Functions Come from an Exact Differential . . . . .	505
6.6.1	Overview . . . . .	505
6.6.2	A Democratic Differential Form . . . . .	505
6.6.3	Information about $\mathcal{M}$ Carried by the Democratic Form . . . . .	507
6.6.4	Breaking the Degeneracy . . . . .	508
6.7	Plethora of Transformation Functions . . . . .	511
6.7.1	Derivation . . . . .	511
6.7.2	Discussion . . . . .	517
6.7.3	Hamilton-Jacobi Theory and Equations . . . . .	521
6.8	Symplectic Invariants . . . . .	533
6.8.1	Liouville's Theorem . . . . .	534
6.8.2	Gromov's Nonsqueezing Theorem and the Symplectic Camel . . . . .	535
6.8.3	Poincaré Integral Invariants . . . . .	539
6.8.4	Connection between Surface and Line Integrals . . . . .	542
6.8.5	Poincaré-Cartan Integral Invariant . . . . .	545
6.9	Poincaré Surface of Section and Poincaré Return Maps . . . . .	552
6.10	Overview and Preview . . . . .	555
<b>7</b>	<b>Lie Transformations and Symplectic Maps</b>	<b>563</b>
7.1	Production of Symplectic Maps . . . . .	563
7.2	Realization of the Group $Sp(2n)$ and Its Subgroups . . . . .	567
7.3	Invariant Scalar Product . . . . .	575
7.4	Symplectic Map for Flow of Time-Independent Hamiltonian . . . . .	586
7.5	Taylor Maps and Jets . . . . .	589
7.6	Factorization Theorem . . . . .	591
7.7	Inclusion of Translations . . . . .	599
7.8	Other Factorizations . . . . .	604
7.9	Coordinates and Connectivity . . . . .	605
7.10	Storage Requirements . . . . .	607
<b>8</b>	<b>A Calculus for Lie Transformations</b>	<b>615</b>
8.1	Adjoint Lie Operators and the Adjoint Lie Algebra . . . . .	615
8.2	Formulas Involving Adjoint Lie Operators . . . . .	617
8.3	Questions of Order and other Miscellaneous Mysteries . . . . .	621
8.4	Lie Concatenation Formulas . . . . .	632
8.5	Map Inversion and Reverse Factorization . . . . .	640
8.6	Taylor and Hybrid Taylor-Lie Concatenation and Inversion . . . . .	641
8.7	Working with Exponents . . . . .	650

8.7.1	Formulas for Combining Exponents . . . . .	650
8.7.2	Nature of Single Exponent Form . . . . .	653
8.8	Zassenhaus or Factorization Formulas . . . . .	655
8.9	Ideals, Quotients, and Gradings . . . . .	657
<b>9</b>	<b>Inclusion of Translations in the Calculus</b>	<b>681</b>
9.1	Introduction . . . . .	681
9.2	The Inhomogeneous Symplectic Group $ISp(2n, R)$ . . . . .	682
9.3	Lie Concatenation in the General Nonlinear Case . . . . .	689
9.4	Canonical Treatment of Translations . . . . .	698
9.4.1	Preliminaries . . . . .	698
9.4.2	Case of Maps with no Nonlinear Part . . . . .	703
9.4.3	Case of General Maps . . . . .	707
9.5	Map Inversion and Reverse and Mixed Factorizations . . . . .	718
9.6	Taylor and Hybrid Taylor-Lie Concatenation and Inversion . . . . .	721
<b>10</b>	<b>Computation of Transfer Maps</b>	<b>729</b>
10.1	Equation of Motion . . . . .	729
10.2	Series Solution . . . . .	731
10.3	Exponential Solution . . . . .	732
10.4	Factored Product Solution . . . . .	735
10.5	Lie Concatenation Revisited . . . . .	743
10.6	Direct Taylor Summation . . . . .	748
10.7	Scaling, Splitting, and Squaring . . . . .	753
10.8	Canonical Treatment of Errors . . . . .	762
10.9	Symplectic Integration and Zassenhaus Formulas . . . . .	767
10.9.1	$T + V$ Splitting . . . . .	768
10.9.2	Motion in General Electromagnetic Fields . . . . .	772
10.10	Zassenhaus Formulas and Map Computation . . . . .	777
10.10.1	Case of $T + V$ or General Electromagnetic Field Hamiltonians . . . . .	777
10.10.2	Case of Hamiltonians Expanded in Homogeneous Polynomials . . . . .	778
10.11	Other Zassenhaus Formulas . . . . .	782
10.12	Taylor Methods and the Complete Variational Equations . . . . .	786
10.12.1	Case of No or Ignored Parameter Dependence . . . . .	788
10.12.2	Inclusion of Parameter Dependence . . . . .	789
10.12.3	Solution of Complete Variational Equations Using Forward Integration . . . . .	790
10.12.4	Application of Forward Integration to the Two-Variable Case . . . . .	791
10.12.5	Solution of Complete Variational Equations Using Backward Integration . . . . .	795
10.12.6	The Two-Variable Case Revisited . . . . .	797
10.12.7	Application to Duffing's Equation . . . . .	799
10.12.8	Expanding in Parameters as Well . . . . .	801
10.12.9	Taylor Methods for the Hamiltonian Case . . . . .	807

<b>11</b>	<b>Transfer Maps for Idealized Straight Beam-Line Elements</b>	<b>817</b>
11.1	Background . . . . .	817
11.2	Drift . . . . .	817
11.3	Quadrupole . . . . .	817
11.4	Axial Rotation . . . . .	817
11.5	Combined Function Quadrupole . . . . .	817
11.6	Rare Earth Cobalt Quadrupole . . . . .	817
11.7	Sextupole . . . . .	817
11.8	Octupole . . . . .	817
11.9	Higher-Order Multipoles . . . . .	817
11.10	Thin Lens Multipoles . . . . .	817
11.11	Solenoid . . . . .	817
11.12	Radio Frequency Cavity . . . . .	819
<b>12</b>	<b>Transfer Maps for Idealized Curved Beam-Line Elements</b>	<b>823</b>
12.1	Background . . . . .	823
12.2	Sector Bend . . . . .	823
12.3	Parallel (Rectangular) Faced Bend . . . . .	823
12.4	Hard-Edge Fringe Fields . . . . .	823
12.5	Pole Face Rotations . . . . .	823
12.6	General Bend . . . . .	823
12.7	Combined Function Bend . . . . .	823
<b>13</b>	<b>Cylindrical Harmonic and Taylor Expansions</b>	<b>825</b>
13.1	Introduction . . . . .	825
13.2	Cylindrical Harmonic Expansion . . . . .	826
13.2.1	Complex Cylindrical Harmonic Expansion . . . . .	827
13.2.2	Real Cylindrical Harmonic Expansion . . . . .	829
13.2.3	Some Simple Examples: $m = 0, 1, 2$ . . . . .	833
13.2.4	Magnetic Field Expansions for the General Case . . . . .	835
13.2.5	Symmetry and Allowed and Forbidden Multipoles . . . . .	837
13.3	Determination of the Vector Potential: Azimuthal-Free Gauge . . . . .	839
13.3.1	Derivation . . . . .	840
13.3.2	Some Simple Examples: $m = 1, 2$ . . . . .	842
13.4	Determination of the Vector Potential: Coulomb Gauge . . . . .	846
13.4.1	The $m = 0$ Case . . . . .	846
13.4.2	The $m \geq 1$ Cases . . . . .	849
13.5	Uniqueness of Coulomb Gauge . . . . .	857
13.6	Need for Generalized Gradients and the Use of Surface Data . . . . .	864
13.7	Magnetic Monopole Doublet Example . . . . .	865
13.7.1	Magnetic Scalar Potential and Magnetic Field . . . . .	865
13.7.2	Analytic On-Axis Gradients for Monopole Doublet . . . . .	869

<b>14</b>	<b>Realistic Transfer Maps for Straight Beam-Line Elements</b>	<b>881</b>
14.1	Introduction . . . . .	881
14.2	Use of Potential Data on Surface of Circular Cylinder . . . . .	882
14.3	Use of Field Data on Surface of Circular Cylinder . . . . .	885
14.4	Use of Field Data on Surface of Elliptical Cylinder . . . . .	888
14.4.1	Background . . . . .	888
14.4.2	Elliptic Coordinates . . . . .	888
14.4.3	Mathieu Equations . . . . .	892
14.4.4	Periodic Mathieu Functions and Separation Constants . . . . .	893
14.4.5	Modified Mathieu Functions . . . . .	908
14.4.6	Analyticity in $x$ and $y$ . . . . .	912
14.4.7	Elliptic Cylinder Harmonic Expansion and On-Axis Gradients . . . . .	912
14.5	Use of Field Data on Surface of Rectangular Cylinder . . . . .	916
14.5.1	Finding the Magnetic Scalar Potential $\psi(x, y, z)$ . . . . .	916
14.5.2	Finding the On-Axis Gradients . . . . .	923
14.5.3	Fourier-Bessel Connection Coefficients . . . . .	925
<b>15</b>	<b>Tools for Numerical Implementation</b>	<b>935</b>
15.1	Third-Order Splines . . . . .	935
15.1.1	Fitting Over an Interval . . . . .	935
15.1.2	Periodic Splines . . . . .	938
15.1.3	Error Estimate for Spline Approximation . . . . .	940
15.2	Interpolation . . . . .	942
15.2.1	Bicubic Interpolation . . . . .	943
15.2.2	Bicubic Spline Interpolation . . . . .	947
15.3	Fourier Transforms . . . . .	948
15.3.1	Exact Fourier Transform and its Large $ k $ Behavior . . . . .	948
15.3.2	Inverse Fourier Transform . . . . .	949
15.3.3	Discrete Fourier Transform . . . . .	953
15.3.4	Discrete Inverse Fourier Transform . . . . .	957
15.3.5	Spline-Based Fourier Transforms . . . . .	957
15.3.6	Fast Spline-Based Fourier Transforms . . . . .	967
15.4	Bessel Functions . . . . .	968
15.5	Mathieu Functions . . . . .	968
15.5.1	Calculation of Separation Constants $a_n(q)$ and $b_n(q)$ . . . . .	968
15.5.2	Calculation of Mathieu Functions . . . . .	968
15.5.3	Calculation of Fourier and Mathieu-Bessel Connection Coefficients . . . . .	970
<b>16</b>	<b>Numerical Benchmarks</b>	<b>975</b>
16.1	Circular Cylinder Numerical Results for Monopole Doublet . . . . .	975
16.1.1	Testing the Spline-Based Inverse ( $k \rightarrow z$ ) Fourier Transform . . . . .	975
16.1.2	Testing the Forward ( $z \rightarrow k$ ) and ( $\phi \rightarrow m$ ) Fourier Transforms . . . . .	982
16.1.3	Test of Interpolation off a Grid . . . . .	986
16.1.4	Reproduction of Interior Field Values . . . . .	988
16.2	Elliptical Cylinder Numerical Results for Monopole Doublet . . . . .	1001

16.2.1	Finding the Mathieu Coefficients . . . . .	1001
16.2.2	Behavior of Kernels . . . . .	1010
16.2.3	Truncation of Series . . . . .	1011
16.2.4	Approximation of Angular Integrals by Riemann Sums . . . . .	1017
16.2.5	Further Tests . . . . .	1026
16.2.6	Completion of Test . . . . .	1026
16.3	Rectangular Cylinder Numerical Results for Monopole Doublet . . . . .	1038
<b>17</b>	<b>Smoothing and Insensitivity to Errors</b>	<b>1041</b>
17.1	Introduction . . . . .	1041
17.2	Circular Cylinders . . . . .	1042
17.3	Elliptic Cylinders . . . . .	1060
17.4	Rectangular Cylinders . . . . .	1084
<b>18</b>	<b>Applications of Cylindrical Surface Methods</b>	<b>1087</b>
18.1	Application to Final Focus Quadrupoles . . . . .	1087
18.2	Application to Solenoids . . . . .	1096
18.3	Application to Wigglers . . . . .	1099
18.4	Application to Radio-Frequency Cavities . . . . .	1099
<b>19</b>	<b>Realistic Transfer Maps for Curved Beam-Line Elements</b>	<b>1103</b>
19.1	Introduction . . . . .	1103
19.2	Mathematical Tools . . . . .	1104
19.3	Numerical Benchmark . . . . .	1104
19.4	Smoothing and Insensitivity to Errors . . . . .	1104
19.5	Application to a Storage-Ring Dipole . . . . .	1104
<b>20</b>	<b>Error Effects and the Euclidean Group</b>	<b>1105</b>
<b>21</b>	<b>Representations of <math>sp(2n)</math> and Related Matters</b>	<b>1107</b>
21.1	Structure of $sp(2, R)$ . . . . .	1108
21.2	Representations of $sp(2, R)$ . . . . .	1110
21.3	Symplectic Classification of Analytic Vector Fields in Two Variables . . . . .	1114
21.4	Structure of $sp(4, R)$ . . . . .	1123
21.5	Representations of $sp(4, R)$ . . . . .	1126
21.6	Symplectic Classification of Analytic Vector Fields in Four Variables . . . . .	1131
21.7	Structure of $sp(6, R)$ . . . . .	1137
21.8	Representations of $sp(6, R)$ . . . . .	1141
21.9	Symplectic Classification of Analytic Vector Fields in Six Variables . . . . .	1147
21.10	Scalar Product and Projection Operators for Vector Fields . . . . .	1151
21.11	Products and Casimir Operators . . . . .	1159
21.11.1	The Quadratic Casimir Operator . . . . .	1160
21.11.2	Applications of the Quadratic Casimir Operator . . . . .	1166
21.11.3	Higher-Order Casimir Operators . . . . .	1170
21.12	The Killing Form . . . . .	1175

21.13	Enveloping Algebra . . . . .	1176
21.14	The Symplectic Lie Algebras $sp(8)$ and Beyond . . . . .	1183
<b>22</b>	<b>General Maps</b>	<b>1189</b>
22.1	Lie Factorization of General Maps . . . . .	1189
22.2	Classification of General Two-Dimensional Quadratic Maps . . . . .	1194
22.3	Lie Factorization of General Two-Dimensional Quadratic Maps . . . . .	1199
22.4	Fixed Points . . . . .	1207
22.5	Poincaré Index . . . . .	1214
22.6	Manifolds, and Homoclinic Points and Tangles . . . . .	1227
22.7	The General Hénon Map . . . . .	1238
22.8	Preliminary Study of General Hénon Map . . . . .	1246
22.8.1	Location, Expansion About, and Nature of Fixed Points . . . . .	1246
22.8.2	Lie Factorization About the First (Hyperbolic) Fixed Point . . . . .	1253
22.8.3	Location and Nature of Second Fixed Point . . . . .	1256
22.8.4	Expansion and Lie Factorization About Second Fixed Point . . . . .	1265
22.9	Period Doubling and Strange Attractors . . . . .	1271
22.9.1	Behavior about Hyperbolic Fixed Point . . . . .	1271
22.9.2	Behavior about Second Fixed Point . . . . .	1271
22.10	Attempts at Integrals . . . . .	1273
22.11	Quadratic Maps in Higher Dimensions . . . . .	1273
22.12	Taylor Approximations to Stroboscopic Duffing Map . . . . .	1273
22.12.1	Saddle-Node Bifurcations . . . . .	1273
22.12.2	Pitchfork Bifurcations . . . . .	1283
22.12.3	Infinite Period-Doubling Cascade and Strange Attractor . . . . .	1287
22.12.4	Remerging after the Cascade . . . . .	1307
22.13	Analytic Properties of Fixed Points and Eigenvalues . . . . .	1314
<b>23</b>	<b>Normal Forms for Maps and Their Applications</b>	<b>1321</b>
23.1	Equivalence Relations . . . . .	1321
23.2	Symplectic Conjugacy of Symplectic Maps . . . . .	1322
23.3	Normal Forms for Maps . . . . .	1322
23.4	Sample Normal Forms . . . . .	1324
23.5	Dynamic Maps Without Translation Factor . . . . .	1325
23.6	Dynamic Maps With Translation Factor . . . . .	1325
23.7	Static Maps Without Translation Factor . . . . .	1325
23.7.1	Preparatory Steps . . . . .	1325
23.8	Static Maps With Translation Factor . . . . .	1331
23.9	Tunes, Phase Advances and Slips, Momentum Compaction, Chromaticities, and Anharmonicities . . . . .	1331
23.10	Courant-Snyder Invariants and Lattice Functions . . . . .	1331
23.11	Analysis of Tracking Data . . . . .	1331

<b>24 Lattice Functions</b>	<b>1335</b>
<b>25 Solved and Unsolved Polynomial Orbit Problems: Invariant Theory</b>	<b>1337</b>
25.1 Introduction . . . . .	1337
25.2 Solved Polynomial Orbit Problems . . . . .	1339
25.2.1 First-Order Polynomials . . . . .	1339
25.2.2 Second-Order Polynomials . . . . .	1340
25.3 Mostly Unsolved Polynomial Orbit Problems . . . . .	1367
25.4 Application to Analytic Properties . . . . .	1369
<b>26 Beam Description and Moment Transport</b>	<b>1383</b>
26.1 Beam Matching . . . . .	1383
26.2 Moments . . . . .	1383
26.3 Emittances and Moment Invariants . . . . .	1383
<b>27 Optimal Evaluation of Symplectic Maps</b>	<b>1385</b>
27.1 Overview of Symplectic Map Approximation . . . . .	1385
27.2 Symplectic Completion of Symplectic Jets . . . . .	1391
27.2.1 Criteria . . . . .	1391
27.2.2 Monomial Approximation . . . . .	1391
27.2.3 Transformation Function Approximation . . . . .	1391
27.2.4 Cremona Maps . . . . .	1391
27.3 Connection Between Transformation Functions and Lie Generators . . . . .	1391
27.3.1 Method of Calculation . . . . .	1392
27.3.2 Computing $g_2$ . . . . .	1394
27.3.3 Low Order Results: Computing $g_3$ and $g_4$ . . . . .	1395
27.3.4 Two Examples . . . . .	1398
27.3.5 Exploration . . . . .	1399
27.3.6 Comments and Comparisons . . . . .	1409
27.4 Use of Poincaré Transformation Function . . . . .	1412
27.4.1 Determination of Poincaré Transformation Function in Terms of $H$ . . . . .	1412
27.4.2 Application to Quadratic Hamiltonian . . . . .	1413
27.4.3 Application to Symplectic Approximation . . . . .	1414
27.5 Use of Other Transformation Functions . . . . .	1416
27.6 Cremona Approximation . . . . .	1416
<b>28 Long-Term Behavior and Dynamic Aperture</b>	<b>1419</b>
<b>29 Reversal Symmetry</b>	<b>1421</b>
29.1 Reversal Operator . . . . .	1421
29.2 Applications . . . . .	1427
29.3 General Consequences for Straight and Circular Machines . . . . .	1435
29.4 Consequences for some Special Cases . . . . .	1440
29.5 Consequences for Closed Orbit in a Circular Machine . . . . .	1441
29.6 Consequences for Courant-Snyder Functions in a Circular Machine . . . . .	1446

29.7	Some Nonlinear Consequences . . . . .	1452
<b>30</b>	<b>Standard First- and Higher-Order Optical Modules</b>	<b>1461</b>
<b>31</b>	<b>Analyticity and Convergence</b>	<b>1463</b>
31.1	Analyticity in One Complex Variable . . . . .	1463
31.2	Analyticity in Several Complex Variables . . . . .	1467
31.3	Convergence of Homogeneous Polynomial Series . . . . .	1479
31.4	Application to Potentials and Fields . . . . .	1488
31.5	Application to Taylor Maps: The Anharmonic Oscillator . . . . .	1488
31.6	Application to Taylor Maps: The Pendulum . . . . .	1488
31.7	Convergence of the BCH Series . . . . .	1488
31.8	Convergence of Lie Transformations and the Factored Product Representation	1488
<b>32</b>	<b>Truncated Power Series Algebra</b>	<b>1493</b>
32.1	Introduction . . . . .	1493
32.2	Monomial Indexing . . . . .	1494
32.3	Scalar Multiplication and Polynomial Addition . . . . .	1510
32.4	Polynomial Multiplication . . . . .	1511
32.5	Look-Up Tables . . . . .	1512
32.6	Scripts . . . . .	1518
32.7	Look-Back Tables . . . . .	1525
32.8	Poisson Bracketing . . . . .	1533
32.9	Linear Map Action . . . . .	1541
32.10	General Vector Fields . . . . .	1544
32.11	Expanding Functions of Polynomials . . . . .	1546
32.12	Differential Algebra . . . . .	1546
32.13	Other Methods . . . . .	1546
<b>A</b>	<b>Størmer-Cowell and Nyström Integration Methods</b>	<b>1549</b>
<b>B</b>	<b>Computer Programs for Numerical Integration</b>	<b>1565</b>
<b>C</b>	<b>Baker-Campbell-Hausdorff and Zassenhaus Formulas</b>	<b>1577</b>
<b>D</b>	<b>Canonical Transformations</b>	<b>1583</b>
<b>E</b>	<b>Mathematica Notebooks</b>	<b>1585</b>
<b>F</b>	<b>Analyticity and Aberration Expansions</b>	<b>1587</b>
<b>G</b>	<b>Invariant Scalar Products</b>	<b>1603</b>
<b>H</b>	<b>Harmonic Functions</b>	<b>1605</b>
H.1	Computation of Transverse Gradients . . . . .	1605
H.2	Range of Transverse Gradient Operators . . . . .	1614
H.3	Harmonic Functions in Two Variables . . . . .	1620

---

<b>I</b>	<b>Poisson Bracket Relations</b>	<b>1623</b>
I.1	Poisson Brackets . . . . .	1623
I.2	Preparatory Results . . . . .	1625
I.3	Application . . . . .	1626
<b>J</b>	<b>Feigenbaum Cascade Denied</b>	<b>1629</b>
<b>K</b>	<b>Supplement to Chapter 14</b>	<b>1637</b>
K.1	Computation of Generalized Gradients from Spinning Coil Data . . . . .	1637
K.2	Computation of Generalized Gradients from Current Data . . . . .	1639
<b>L</b>	<b>Spline Routines</b>	<b>1643</b>
<b>M</b>	<b>Routines for Mathieu Separation Constants <math>a_n(q)</math> and <math>b_n(q)</math></b>	<b>1651</b>
<b>N</b>	<b>Mathieu-Bessel Connection Coefficients</b>	<b>1661</b>
<b>O</b>	<b>Quadratic Forms</b>	<b>1665</b>