Homework 1 Solutions

Note on homework solutions: The solutions I give are not complete in all cases, but they are supposed to give the main ideas. Your solutions should be completely explicit!

1. (a) Compute

$$\frac{\partial}{\partial t_f}\hat{U}(t_f, t_i) = \frac{\partial}{\partial t_f}\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{t_i}^{t_f} dt_1 \cdots \int_{t_i}^{t_f} dt_n \, T\left[\hat{H}(t_1)\cdots\hat{H}(t_n)\right]. \tag{1}$$

The derivative can act on one of the n limits of integration t_f. In each such term, we get a factor of $\hat{H}(t_f)$, which the time ordering sends all the way to the left. Therefore, all n such terms are identical, and we get

$$\begin{aligned}\frac{\partial}{\partial t_f}\hat{U}(t_f, t_i) &= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\, n\hat{H}(t_f)\int_{t_i}^{t_f} dt_2 \cdots \int_{t_i}^{t_f} dt_n \, T\left[\hat{H}(t_2)\cdots\hat{H}(t_n)\right] &(2)\\ &= -i\hat{H}(t_f)\hat{U}(t_f, t_i). &(3)\end{aligned}$$

This is the Schrödinger equation for the time evolution operator. Note also that it obeys the initial condition

$$\hat{U}(t_i, t_i) = 1. \tag{4}$$

These are the defining properties of the time evolution operator.

(b) Compute

$$\frac{\partial}{\partial t}\left[\hat{U}(t_f, t)\hat{U}(t, t_i)\right] = \left[\hat{U}(t_f, t) i\hat{H}(t_i)\right]\hat{U}(t, t_i) + \hat{U}(t_f, t)\left[-i\hat{H}(t_i)\hat{U}(t, t_i)\right] = 0. \tag{5}$$

Therefore, the left-hand side is independent of t. As $t \to t_i$ or $t \to t_f$ the identity is clearly true. Therefore it is true for all t.

(c), (d) The derivation follows exactly the usual steps. There are no subtleties.

3. (a) The energy eigenstates are the momentum eigenstates, so a general solution can be written

$$\psi(x, t) = \int_{-\infty}^{\infty} dp\, e^{ipx}\tilde{\psi}(p)e^{-i\sqrt{p^2+m^2}\,t}. \tag{6}$$

Demanding that $\psi(x, t = 0) = \delta(x)$ gives $\tilde{\psi}(p) = \text{constant}$, so we have

$$\psi(x, t) = N\int_{-\infty}^{\infty} dp\, e^{ipx}\, e^{-i\sqrt{p^2+m^2}\,t}, \tag{7}$$

where N is a normalization factor. Near $t = 0$, the wavefunction is dominated by small x, hence large p. We can therefore approximate

$$\sqrt{p^2 + m^2} \simeq |p|. \tag{8}$$

We obtain

$$\psi(x,t) = N \int_{-\infty}^{\infty} dp\, e^{ipx}\, e^{-i|p|t} = f(t-x) + f(t+x), \tag{9}$$

where

$$f(t) = N \int_{-\infty}^{\infty} e^{-ipt}\theta(p). \tag{10}$$

is the Fourier transform of the θ function. This is given by the identity

$$\theta(p) = -\frac{i}{2\pi} \int_{-\infty}^{\infty} dt\, e^{ipt}\, \frac{1}{t - i\epsilon}, \tag{11}$$

where $\epsilon \to 0+$. We therefore have

$$f(t) \propto \frac{1}{t}, \tag{12}$$

which definitely gives a wavefunction that is non-vanishing outside the light cone.

(b) when the width is larger than $1/m$, the momenta are smaller than m, and we can approximate

$$\sqrt{p^2 + m^2} = m + \frac{p^2}{2m} + \cdots \tag{13}$$

This is the usual non-relativistic quantum mechanics limit, and the speed of the wavepacket spreading is of order

$$v \sim \frac{p}{m} \sim \frac{1}{m\Delta x} \ll 1. \tag{14}$$

(c) Expanding

$$\hat{H} = \sqrt{\hat{p}^2 + m^2} = m + \frac{\hat{p}^2}{2m} - \frac{\hat{p}^4}{8m^3} + \mathcal{O}(\hat{p}^6). \tag{15}$$