Homework 3

due 2/22/07

1. Consider a theory of a real scalar field ϕ in $3+1$ dimensions, where we impose that the Lagrangian is invariant under the shift symmetry

$$\phi \mapsto \phi + \lambda, \tag{1}$$

where λ is an arbitrary constant. This symmetry means that the Lagrangian can depend only on derivatives of ϕ. In particular, the symmetry forbids a mass term for ϕ, so this gives a theory with a scalar that is naturally massless.

(a) When writing down the possible terms in the Lagrangian, we can simplify them by using integration by parts. That is, we can neglect any interaction that is a total derivative

$$\Delta\mathcal{L} = \partial_\mu X^\mu, \tag{2}$$

where X^μ is any function of the fields. Formally, this is because any such term gives a surface term in the action when integrated over all spacetime. Give a less formal reason for neglecting such terms by showing that the Feynman rule arising from any such term is identically zero.

(b) Find the lowest-dimension interaction term(s) allowed by the symmetry, eliminating redundant terms by integration by parts. Introduce a coupling constant for each such term. Use these Feynman rules to find the lowest order (tree-level) 2-to-2 scattering amplitude $\mathcal{M}$ for this theory as a function of the coupling(s) and the Mandelstam variables

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2. \tag{3}$$

where $p_1, \ldots, p_4$ are the external momenta.

(c) Find the interaction term(s) with the next-lowest dimension allowed by the symmetry, and introduce new coupling constants for each one. Find the leading tree-level contribution to the 2-to-2 scattering amplitude arising from the new coupling(s).

(d) Obviously, we could keep adding more interactions. When do we stop? Assume that the size of each of the couplings you have introduced above is determined by a mass scale M, *i.e.* each coupling is given by

$$g_i = c_i M^{d_i}, \tag{4}$$

where d_i is the mass dimension of the coupling g_i, and c_i is a pure number of order 1. Suppose that M is much larger than the energies available in our experiment, *i.e.* assume

$$s \sim t \ll M^2. \tag{5}$$

Does the contribution in part (b) or (d) dominate in this limit? Now suppose we include arbitrary additional interaction terms in the Lagrangian involving higher powers of ϕ and derivatives. Show that these give small corrections to the scattering amplitude at tree level, and estimate the fractional correction as a power of s/M^2.

You have therefore derived a 'low energy theorem' for the scattering of a scalar with a shift symmetry (at least at tree level). That is, the complete scattering amplitude is determined by a small number of couplings that have to be determined from experiment. (The same logic will be used to derive a low-energy theorem for the scattering of Goldstone bosons later in the course.)

2. Consider a theory with N real scalar fields ϕ_a ($a = 1, \ldots, N$) in $3 + 1$ dimensions, with Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\partial^\mu \phi_a \partial_\mu \phi_a - \tfrac{1}{2} m^2 \phi_a \phi_a - \frac{\lambda}{8}(\phi_a \phi_a)(\phi_b \phi_b), \tag{6}$$

where repeated indices are summed. Note that the Lagrangian is invariant under

$$\phi_a \mapsto R_{ab}\phi_b, \tag{7}$$

where R is an $N \times N$ matrix satisfying $R^T = R^{-1}$. (That is, R is an element of $O(N)$.) In this problem, you will study this theory in the large-N limit $N \to \infty$.

(a) Write the Feynman rules for this theory. Be sure that you can keep track of the index structure.

(b) Write the diagrams for the 1- and 2-loop contributions to the 1PI 2-point function and the 4-point scattering amplitude. Don't evaluate them, but find the dependence of each diagram on λ and N. Show that we can get a well-defined large-N limit if we take

$$\lambda = \frac{\tilde{\lambda}}{N}, \tag{8}$$

and keep $\tilde{\lambda}$ fixed as we take $N \to \infty$. Argue that this limit is well-defined to all orders in the loop expansion.

(c) Classify the diagrams that contribute to the 1PI 2-point function in the $N \to \infty$ limit. Note that these are 'tadpole' graphs in the sense that they are independent of the external momenta. Write a 'tadpole condition' that determines the physical mass. Show that when the tadpole condition is satisfied, the tree-level 2-point function is exact, so the mass m in the propagator is the same as the physical mass.

(d) Classify all diagrams that contribute to the 4-point function in the $N \to \infty$ limit. Show that the infinite number of diagrams organize themselves into a geometric series. Sum the series. Note that this calculation is valid for any value of $\tilde{\lambda}$, including strong coupling!

(e) Show that this theory has a bound state by showing that it has a pole in the s channel of the scattering amplitude. Show that the mass of this bound state is approximately given by

$$m_{\text{bound}} = 2m \left[1 - 2e^{-32\pi m/\tilde{\lambda}} \right] \tag{9}$$

for $\tilde{\lambda} \ll m$. The the binding energy of the bound state is therefore very small in this limit. Plot the mass of the bound state as a function of $\tilde{\lambda}/m$ and show that the binding energy is large ($2m - m_{\text{bound}} \gtrsim m$) for $\tilde{\lambda}/m \gtrsim 1$.