Homework 2

due 2/13/07

1. Consider the $0+0$ dimensional 'Euclidean' quantum field theory, defined by the ordinary integral

$$Z(J) = \int_{-\infty}^{\infty} d\phi \, e^{-S(\phi)+J\phi}, \tag{1}$$

with

$$S(\phi) = \tfrac{1}{2}\phi^2 + \frac{\lambda}{4!}\phi^4. \tag{2}$$

This theory is not physically very interesting, but it does have the same diagrammatic expansion as ϕ^4 theory in higher dimensions, and therefore can give us information about the terms in the expansion.

(a) Write the Feynman rules for the expansion of the correlation functions

$$\langle \phi^n \rangle = \frac{\int d\phi \, e^{-S(\phi)} \phi^n}{\int d\phi \, e^{-S(\phi)}} \tag{3}$$

in powers of λ. Show that the coefficient of each term in the expansion is equal to the sum of the symmetry factors of all of the diagrams that contribute at that order. Since the symmetry factors of the diagrams are the same in any spacetime dimension (and in Minkowski space), this gives a 'checksum' for the symmetry factors that can be a useful check for high-order calculations in larger spacetime dimensions.

(b) Compute the expansion of the 2-point function to $\mathcal{O}(\lambda^3)$ and the 4-point function to $\mathcal{O}(\lambda^2)$. Do the calculation in two different ways. First, compute it directly by explicitly expanding the integral. Second, write the diagrammatic expansion and compute the symmetry factors of all of the diagrams that contribute. Check that the answers agree.

(c) Show that the theory has a well-defined *strong coupling expansion* in inverse powers of λ. Compute the leading and subleading terms in the 2-point function and the 4-point function in this expansion.

(d) Suppose that $\lambda = 0.2$. Compute enough terms in the weak coupling expansion of the 2-point function that you can start to see clearly numerically that the series diverges. (I recommend you use Mathematica or other symbolic computation software.) Suppose we did not have access to the exact answer. Then we could guess that the best series approximation is obtained by keeping successive terms in the series as long as they are decreasing, and estimate the order of magnitude of the error by the first term that we throw out. Check how well this guess works for $\lambda = 0.2$ by computing the exact answer (numerically, if necessary). See how well your results for the coefficients agree with the asymptotic estimates for the coefficients derived in the notes.

2. Consider ϕ^4 theory in d dimensions, defined by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{4}$$

(a) Determine the mass dimension of m and λ for arbitrary d. For what values of d is the dimension of m and λ positive?

The signifiance of a coupling c having positive mass dimension is the following. Suppose we try to treat c as a perturbation, *i.e.* we expand physical quantities in powers of c. Schematically, the expansion looks like

$$\text{physical quantity} = (c \to 0 \text{ limit}) \left[1 + \mathcal{O}\left(\frac{c}{p^{\dim(c)}}\right)\right]. \tag{5}$$

If $\dim(c) > 0$, this expansion converges at large p (short distance), but fails for small p (large distance). Therefore, the coupling is effectively weak at short distances and strong at long distances. A coupling with positive mass dimension is called a *relevant coupling.*

(b) Suppose we are interested in the long-distance behavior for spacetime dimensions d where λ is a relevant coupling. Then we need an expansion in inverse powers of λ, *i.e.* a strong coupling expansion. To write the strong coupling expansion, start with the massless Euclidean ϕ^4 theory on a lattice, defined by the action

$$S = \sum_x a^d \left[\tfrac{1}{2}\sum_\mu \left(\frac{\phi_{x+\mu} - \phi_x}{a}\right)^2 + \frac{\lambda}{4!}\phi_x\right] \tag{6}$$

Here x are lattice points, a is the lattice spacing, and μ runs over unit lattice vectors. We set $m = 0$ so that there is only one relevant coupling in the problem. Show that the generating functional can be written in the form

$$Z[J] = \int \left[\prod_x \left(d\phi_x\, e^{-f(\phi_x)}\right)\right] \exp\left\{-S_{\text{hop}}[\phi] + \sum_x a^d J_x \phi_x\right\}, \tag{7}$$

where

$$S_{\text{hop}} \propto \sum_x \sum_\mu \phi_{x+\mu}\phi_x. \tag{8}$$

Find the precise forms of f and S_{hop}. By rescaling the fields, argue that the strong coupling expansion is an expansion in powers of S_{hop}. Because S_{hop} connects neighboring points, it is called a *hopping term*, and the expansion in powers of S_{hop} is called the *hopping expansion.*

(c) In this part of the problem, you will compute the leading contribution to the 2-point function $\langle \phi(x)\phi(y) \rangle$ in the strong coupling expansion. To make the calculation as easy as possible, choose the points x and y to lie N lattice spacings apart along one of the lattice directions. Note that because $f(\phi)$ is an even function, we have

$$\int d\phi_x \, \phi_x^n e^{-f(\phi_x)} = 0 \tag{9}$$

for $n =$ odd. Show that this implies that the leading contribution to the propagator comes from a term with N powers of S_{hop} connecting the initial and final point. Compute this contribution, carefully keeping track of all factors of N.

(d) Express your answer in terms of the physical distance $r = |y - x| = aN$. Now try to take the continuum limit $a \to 0$ with r held fixed. A well-defined limit of the 2-point function would be a result of the form

$$\langle \phi(x)\phi(y) \rangle = Z\Delta(r), \tag{10}$$

where Z is a possibly divergent constant (independent of r), and $\Delta(r)$ is a well-defined finite function of r. For what values of d (if any) can you take the continuum limit?