Homework 1

due 2/6/07

1. In this problem you will derive the path integral for a quantum-mechanical system with a potential that depends explicitly on time:

$$\hat{H}(t) = \frac{\hat{p}^2}{2m} + V(\hat{q}, t). \tag{1}$$

(a) Show that the time evolution operator for this system is given by the time-ordered exponential

$$\begin{aligned} \hat{U}(t_f, t_i) &= \text{Texp}\left\{-i \int_{t_i}^{t_f} dt\, \hat{H}(t)\right\} \\ &\equiv \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_i}^{t_f} dt_1 \cdots \int_{t_i}^{t_f} dt_n\, T\left[\hat{H}(t_1) \cdots \hat{H}(t_n)\right], \end{aligned} \tag{2}$$

where the T symbol in the last line puts the operators in order of decreasing time ('later to the left').

(b) Use the definition of the time-ordered exponential to show that

$$\hat{U}(t_f, t_i) = \hat{U}(t_f, t)\hat{U}(t, t_i). \tag{3}$$

Is this true even if $t < t_i$ or $t > t_f$? What if $t_f < t_i$?

(c) Derive the Hamiltonian version of the path integral for this system by splitting the time interval from t_i to t_f into small time intervals Δt using Eq. (3).

(d) The Lagrangian for a particle of charge e moving in an electromagnetic field with potentials $\Phi(\vec{x}, t)$ and $\vec{A}(\vec{x}, t)$ is

$$L = \tfrac{1}{2} m \vec{v}^{\,2} - e\Phi + \frac{e}{c} \vec{v} \cdot \vec{A}. \tag{4}$$

Use the result above to write the Lagrangian path integral for this theory, treating Φ and $\vec{A}$ as classical background fields. Do you get the naïvely expected result?

3. Consider the Hamiltonian

$$\hat{H} = +\sqrt{\hat{p}^2 + m^2}. \tag{5}$$

(Remember we are using units where $c = 1$, $\hbar = 1$.) This describes a free particle in one dimension with a relativistic dispersion relation. The square root is defined to be the positive root, which makes sense because $\hat{p}^2 + m^2$ is a positive operator.

(a) Consider a wavefunction that is localized at $x = 0$ at $t = 0$:

$$\Psi(x, t = 0) = \delta(x). \tag{6}$$

Find the wavefunction for $t > 0$. Show that the wavefunction spreads at a speed faster than the speed of light. If this is the theory of nature, it seems that we can use this to send a signal faster than light!

(b) Use your solution above to show that once the wavefunction has a width $\Delta x \gg 1/m$, the spreading of the wavefunction is slower than the speed of light. This suggests that a fundamental limitation on the localization of a particle in position space can resolve the problem with faster-than-light communication in relativistic quantum mechanics.

(c) For wavefunctions with $\Delta x \gg 1/m$, argue that we can expand the Hamiltonian in powers of $\hat{p}^2$. Find the Hamiltonian including terms up to $\mathcal{O}(\hat{p}^4)$.

(d) Use the effective Hamiltonian above to answer the following question. Consider a relativistic harmonic oscillator with Hamiltonian

$$H = \sqrt{\hat{p}^2 + m^2} + \frac{\kappa}{2}\hat{x}^2. \tag{7}$$

Find the leading relativistic correction to the ground state energy. What is the condition on κ and m for this to be a good approximation? Explain this condition using the uncertainty relation.