PHYS 752 (Spring 2022): Elementary Particle Physics II: Theory

Homeworks

General guidelines for homework

(1). Please read the problem *number* (when it is from the textbook by Lahiri and Pal, sometimes abbreviated as *LP*) correctly.

In particular, note that the number to left of decimal point refers to the relevant chapter and the one to the *right* is the actual problem number.

Finally, please make sure you have the *second* edition of LP.

(2). Many of the following homework problems have *multiple* parts. So, it is your responsibility to read the *full* statement of the problem carefully.

(3). Similarly, hints/suggestions/instructions are provided for many problems, so make use of them.
(4). Most (if not all!) of the problems here are *not* mathematical ones, i.e., are instead (almost purely) *physics*-based. So, *unless it is (explicitly) stated otherwise*, feel *free* to use any computer programs (such as mathematica) for this solving them or to look up (purely) mathematical formulae in reference tables (including online). However, if you take such a step, please indicate what exactly you did here (this will help me grade your work appropriately).

(5). You are of course welcome to ask for help (for example, hints) on homework from the instructor. Also, "limited" discussion with other students (including those *not* in the class) *is* allowed (and encouraged): for example, just in order to get started, but the actual problem-solving part should be your own work.

(6). In order to get full credit for homework (*and* exam) problems, you should show as many steps as you can.

(7). All homeworks are to be uploaded (preferably as a PDF file) onto the ELMS course webpage.

1 Homework 1 [Renormalization in QED (I)], due Monday, February 14 on ELMS

1.1 Ward-Takahashi identity

Exercise 12.2 of Lahiri and Pal.

The Ward-Takahashi identity was proved at 1-loop for (pure) QED in lecture (and in section 12.4 of Lahiri and Pal). The idea in this problem is to similarly prove it for the *combination* of Yukawa theory and QED.

Just to be clear, for this problem, $\Gamma_\mu^{(1)}$ on left-hand side (LHS) of Eq. 12.20 of Lahiri and Pal is the 1-loop correction from Yukawa coupling to the electron-photon vertex and similarly, $\Sigma^{(1)}$ on right-hand side (RHS) is the 1-loop correction from the Yukawa coupling to the electron self-energy.

1.2 Fermion self-energy

The calculation of fermion self-energy is performed using *Pauli-Villars* regularization in sections 12.6.2 of Lahiri and Pal and adding counterterms in this scheme is (partially) discussed in section 12.7.2. The idea here is to first *complete* the discussion of counterterms in the same scheme (problem 1.2.1 below).

Then (problems 1.2.2 and 1.2.3), you are to do an analogous procedure using *dimensional* regularization instead, i.e., same scheme as used for the calculation of vacuum polarization in lecture (and section 12.6.1 of Lahiri and Pal).

Finally, the idea is to perform a *comparison* of the full two-point function in the two schemes (problem 1.2.3).

1.2.1 Counterterms in *Pauli-Villars* regularization

Exercise 12.6 of Lahiri and Pal.

Notes/hints: first, derive Eq. 12.86 from Eq. 12.85: is this result valid for *any* regularization scheme?

The 1-loop result for Pauli-Villars regularization is given in Eq. 12.67. Using this in Eq. 12.86, you are supposed to obtain Eq. 12.87.

Finally, one of the features of the full two-point function in the Pauli-Villars reguralization that you are supposed to calculate in the 2nd part of this exercise is indicated just below Eq. 12.87 of Lahiri and Pal (namely, *in*dependence of the regularization parameter M).

1.2.2 Calculation of 1-loop diagram in *dimensional* regularization

Exercise 12.5 of Lahiri and Pal.

Notes: you can (simply) use formulae in Eq. 12.57 of Lahiri and Pal as stated in the hint for this problem, i.e., *no* need to *prove* them (which is Exercise 12.4 in Lahiri and Pal).

1.2.3 Adding counterterms in *dimensional* regularization and *comparison* with Pauli-Villars regularization

Exercise 12.7 of Lahiri and Pal.

Notes/hints: as already suggested (and as you should have shown) above in problem 1.2.1, Eq. 12.85 gives Eq. 12.86 for dimensional regularization as well.

Also, in the last part, you are supposed to compare to the result of 2nd part of the problem 1.2.1.

Finally, here (and in other similar situations), you do not *have* to actually compute the Feynman parameter integrals *in principle*; for example, it is obviously sufficient to show (up on suitable manipulations) that the *integrands* are equal.

2 Homework 2 [Renormalization in QED (II)], due Monday, March 7 on ELMS

2.1 Relation between coefficients of counterterms for fermion self-energy and vertex diagrams

[This is a (very) tedious/long problem, so do not be "disappointed" if you are not able to complete it. However, it brings across an important physics issue, namely, gauge invariance, and so it is well worth the effort! I have given explanations (and some hints) in the statement of the problem in order to guide you. So, please read the problem carefully.

Also, you should simply use the relevant results obtained in Lahiri and Pal, i.e., you do *not* need to show their derivation again. So, you might have to read *on your own* sections 11.1-11.4 of Lahiri and Pal in order to do this problem (of course, you should ask me if something is not clear).

You can use mathematica to do the Feynman parameter integrals in this problem, although it is not too difficult to them "by hand".

Use the 't Hooft-Feynman gauge, i.e., $\xi = 1$ for the photon propagator.

Finally, note that we will follow the notation of Lahiri and Pal so that number of spacetime dimensions $= 4 - 2\varepsilon$, whereas Peskin and Schroeder use $4 - \varepsilon$ instead. And, Lahiri and Pal assume $e > 0$ vs. Peskin and Schroeder's $e < 0$.]

2.1.1 Coefficient of vertex diagram counterterm

Calculate the coefficient of the vertex diagram counterterm, namely,

$$Z_1 - 1 = F_1^{\text{loop}}(0) \tag{1}$$

where F_1 is the electromagnetic form factor discussed chapter 11 of Lahiri and Pal.

[It is clear from above formula that Z_1 is related to the vertex function, Γ_μ, eveluated for *on*-shell external fermions, even though, in general, the vertex function is defined for off-shell fermions as well.]

In your result, keep the divergent pieces (both UV and IR: see below) and *finite terms only, i.e.,* neglect *terms which vanish as the regulators – whether ε of dimensional regularization or "mass" for photon, m_γ – are set to their physical values, i.e., zero.*

You can simply *re-use* the calculation done in sections 11.3 and 11.4 of Lahiri and Pal, but with the following *modifications*:

"When" to perform Feynman parameter integral: it might be more convenient (but of course it is up to you how to do it actually!) in what follows to *not* perform the integral over the Feynman parameter ζ_3 as in Eq. 11.47 of Lahiri and Pal. Rather, just continue the manipulations from Eq. 11.48 onwards (of course with suitable modifications for dimensional regularization and regulating IR divergence: see below). When you get to the (modified) Eq. 11.80, use the δ-function of Feynman parameters in Eq. 11.46 of Lahiri and Pal in order to integrate over the Feynman parameters $\zeta_{1,2}$ (again instead of eliminating ζ_3 as is done in Lahiri and Pal).

UV-finite, but IR-divergent term: the third term in Eq. 11.81 of Lahiri and Pal (originating, in turn, from the last three terms of Eq. 11.50 of Lahiri and Pal) gives a UV finite contribution to $F_1(0)$ (since $q^2 = 0$, the second term in Eq. 11.81 does not contribute), i.e., it does not diverge as the loop momentum becomes large. So, *there is* no *need to use (UV) regularization in order to calculate this part.*

However, based on simple dimensional analysis/power counting, we *expect* a divergence from arbitrarily small loop momentum (i.e., of the internal photon) in this contribution. This *IR*-divergence actually manisfests as (part of) the intergal over Feynman parameters being divergent (*after* the loop momentum integral has been performed).

So, add a (infinitesimal) photon "mass" to regulate this divergence, i.e., the photon propagator is modified as follows:

$$\frac{g_{\mu\nu}}{k^2} \quad \rightarrow \quad \frac{g_{\mu\nu}}{k^2 - m_\gamma^2} \tag{2}$$

Of course, a photon mass breaks gauge invariance, but it turns out it does so in a "safe" manner, i.e., not spoiling the relations enforced by gauge invariance – you will perform one check of this claim below.

Just like for the UV divergence, it suffices to use a non-zero photon mass only for IR-divergent part of the expression (again, the one gives a divergent integral over Feynman parameters in absence of such a regulator).

[The "kosher" way to regulate this IR-divergence is to add bremsstrahlung diagram, i.e., include emission of a *real* photon from one of the fermion lines. The amplitude for this diagram diverges as momentum of the real photon becomes arbitrarily small. However, in this limit, the extra photon cannot be resolved in a real detector so that the cross-section for this process[1] has to be added to the process without the extra photon emission (that we are calculating here at the loop level). One can show that the IR divergences cancel between these two contributions: see section 12.9.3 of Lahiri and Pal or chapter 6 of Peskin and Schroeder for more details. In this sense, roughly, one can think of the photon mass used as an IR- "regulator" above as corresponding to the smallest photon energy which can be resolved by the detector.]

The UV divergent contribution: it comes from the first term in Eq. 11.50 of Lahiri

[1]To be precise, imagine that the external photon in the vertex diagram corresponds to the static field of a heavy nucleus, as discussed in section 9.9 of Lahiri and Pal.

and Pal, giving the first term in Eq. 11.81 of Lahiri and Pal. Depending on how this is regularized, the problem splits into the following two **parts**:

(i) Use *dimensional* regularization (which preserves gauge invariance, as we saw in the calculation of the vacuum polarization diagram) to evaluate this contribution. You can check that there is no IR divergence in this UV-divergent contribution (based on dimensional analysis/power counting or from the explicit calculation below) so that no need to use a non-zero photon mass here [unlike in part (ii) above].

However, some of the four-dimensional formulae used in going from Eq. 11.50 to Eq. 11.81 have to be modified in order to go to $(4-2\varepsilon)$ dimensions as follows. First, the contraction of γ-matrices becomes:

$$\gamma_\lambda \gamma_\mu \gamma^\lambda = (-2+2\varepsilon)\gamma_\mu \quad \text{(as in Eq. 12.57 in Lahiri and Pal)} \tag{3}$$

$$\gamma_\lambda \gamma_\mu \gamma_\nu \gamma_\rho \gamma^\lambda = (-2+2\varepsilon)\gamma_\rho \gamma_\nu \gamma_\mu \text{ (assuming } \mu \text{ and } \rho \text{ are symmetrized)} \tag{4}$$

Secondly, in $4-2\epsilon$ dimensions, we have

$$\int d^{4-2\varepsilon}k\, k_\mu k_\nu f\left(k^2\right) = \int d^{4-2\varepsilon}k \frac{k^2}{4-2\varepsilon} f\left(k^2\right) g_{\mu\nu} \tag{5}$$

On a different note, you can use simply Eq. A.53 of Lahiri and Pal in order to perform the momentum integral. However, note that some of the *finite* Γ functions appearing in Eq. A.53 of Lahiri and Pal have to be expanded to $O(\varepsilon)$.

Note that the above three $O(\varepsilon)$ "corrrections" have to be included since they will give $O\left(\varepsilon^0\right)$, i.e., finite, terms when multiplied by the divergent, i.e., $O\left(\varepsilon^{-1}\right)$, part coming from the momentum integral.

(ii) Use *cut-off* regularization instead to calculate Z_1, i.e., stick to four dimensions (just like in Eq. 11.81 of Lahiri and Pal), but restrict (loop momentum)$^2 \leq \Lambda^2_{UV}$. Regulate the IR divergence using a photon mass (as before).

2.1.2 Counterterm for fermion self-energy

(i) You should have already calculated $Z_2 - 1$:

$$\begin{aligned} Z_2 - 1 &= -a_{CT} \\ &= a\left(m^2\right) + 2m^2 a'\left(m^2\right) + 2mb'\left(m^2\right) \quad \text{(using the notation of Lahiri and Pal)} \end{aligned} \tag{6}$$

in the problem 1.2 above (using *dimensional* regularization), but perhaps you left the the Feynman parameter integral as is (i.e., just like Eq. 12.87 in Lahiri and Pal which is for Pauli-Villars regularization).

[Again, it is clear that Z_2 is related to the fermion self-energy, Σ, (and its derivatives) evaluated for *on*-shell external fermions, although Σ is (in general) defined for off-shell fermions as well.]

However, just like for vertex diagram, you should find (part of) this Feynman parameter integral to be divergent, reflecting the photon momentum being allowed to be arbitrarily *small* (see discussion in second paragraph below Eq. 12.87 in Lahiri and Pal). So, use a non-zero photom mass in order to regulate this IR divergence (just as in the vertex diagram above).

(ii) Calculate the fermion self-energy diagram and hence Z_2 using *cut-off* regularization (for UV divergence) and photon mass for regulating the IR divergence.

2.1.3 Check of gauge invariance

Show that

(i) $Z_1 = Z_2$ for dimensional regularization

[Again, include only the UV and IR divergences and *finite* terms here, i.e., you can simply drop pieces which vanish in the limit $\varepsilon \to 0$ (i.e., in four dimensions) or photon mass $\to 0$.]

Recall that the Feynman parameters is simply a (very useful!) "trick" to calculate the loop diagram, i.e., there's no "physics" associated with it. So, strictly speaking, $Z_1 = Z_2$ is guaranteed only *after* doing the Feynman parameter integral (i.e., after the trick has been used), i.e., the integrands in the Feynman parameter integral in Z_1 do *not* have to "look" the same as in Z_2.

(ii) However, show that $Z_1 \neq Z_2$ and that they differ by a *finite* term for cut-off regularization.

Recall that this equality was expected in dimensional regularization, based on the Ward-Takahashi identity relating the amplitudes of the fermion self-energy and vertex diagrams which, in turn, follows from gauge invariance/current conservation *and* the fact that this regularization preserves gauge invariance.

However, when we used the cut-off regularization (for UV divergence) instead of dimensional regularization, then we do not find the above equality (since this regulator breaks gauge invariance).

Finally, the above equality (for dimensional regularization) is *not* spoiled by the use of photon mass as IR-regulator which suggests that this regulator is "kosher" – we might comment on this point later when we discuss how to give gauge bosons a mass based on the Higgs mechanism.

2.2 Running of electric charge

Exercise 12.8 of Lahiri and Pal.

[Assume $\alpha(0) = 1/137.03$ as indicated below Eq. 12.111 in Lahiri and Pal.]

Remember to include *color* factor and *fractional* electric charges for quarks.

3 Homework 3 [Spontaneous (Global) Symmetry Breaking], due Monday, March 14 on ELMS

3.1 Massive gauge boson propagator

Exercise 8.1 of Lahiri and Pal.

Derive it along the lines of section 8.2, i.e., as a (classical) Green's function (you might have encountered an analogous derivation in Phys624 – if not, you might have to read carefully section 8.2 on your own).

(Note that, unlike the case of massless gauge boson, there is no need to add a "gauge fixing" term to the Lagrangian in order to define the propagator.)

The above propagator was used in lecture in order to (i) show how gauge boson exchange can generate four-fermion interaction at low energies and (ii) to discuss (non)-renormalizability of this theory (see also next problem).

3.2 Superficial degree of divergence with massive gauge boson

Exercise 12.1 of Lahiri and Pal.

Hint: think about how the relation in 1st line of Eq. 12.3 of Lahiri and Pal is derived and thus whether one can actually write two *separate* such equations, one each for scalar and gauge boson/vector lines.

[In lecture, we "verified" the above (superficial) degree of divergence using a *one*-loop diagram leading to fermion-fermion scattering.]

[It is assumed here that there are *only* massive gauge bosons (whose number of lines is denoted by E_A and a_i) in the theory. However, it should be easy (as follows) to extend this formula for the case when there are *also* mass*less* gauge bosons present: since massless gauge boson lines are on same "footing" as scalar lines (as far as this argument is concerned), E_ϕ then simply denotes the number of external massless gauge boson *and* scalar lines.]

3.3 Nambu-Goldstone boson interactions

Exercise 13.14 of Lahiri and Pal.

This problem shows an example of the general point that Nambu-Goldstone boson interactions vanish in the limit of momentum of any external Nambu-Goldstone boson going to zero; another one is worked out in Fig. 13.2, Eq. 13.64 of Lahiri and Pal.

You can use either linear or radial representation here. In fact, for "extra credit", you can do it both ways and show that they give the same result.

3.4 Representation-independence

Exercise 13.15 of Lahiri and Pal.

Note: the vanishing of this amplitude at low energies is already shown using *linear* representation in Fig. 13.2, Eq. 13.64 of Lahiri and Pal (as indicated in the previous problem). So, the goal here is to demonstrate that we get same (full) result using radial representation instead, i.e., not just the vanishing of amplitude as momenta become small, but the detailed factors associated with it.

This problem can be *tedious* (so feel free to use mathematica etc. in order to simplify algebra), but it is useful since it illustrates the property of "representation-independence" of the amplitudes which was mentioned in lecture (and of course again shows that Nambu-Goldstone boson interactions vanish in the limit of vanishing momentum of any external Nambu-Goldstone boson).

Note: there's a typo in Lahiri and Pal's Eq. 13.64: see
http://www.saha.ac.in/theory/palashbaran.pal/books/qft/2ed/err2.html

Some *suggestions* for this problem: as might have been discussed in Phys624, it is difficult to derive Feynman rules for derivative interactions using canonical quantization (again, due to expression for conjugate momentum of field being now modified due to such an interaction). So, you might not have seen these rules so far.

In spite of this situation, for this problem, just "guess" these rules based on the guideline that each derivative gives one power of momentum of the relevant particle: see, for example, Feynman rule for similar vertex in scalar QED in Peskin and Schroeder problem 9.1 (a) on page 312 or Cheng and Li page 505 of appendix (of course, the rule was actually *derived* in these references!) or page 1 of notes from Phys624 that I taught in Fall 2010 at
http://www.physics.umd.edu/courses/Phys624/agashe/F10/notes/scalarQED.pdf.
(This comment applies to HW 3.3 as well, but only *if* you use radial representation for it).

Also, in showing that these two amplitudes are equal, you might find it useful to use the kinematical relations given just below Eq. 13.64 of Lahiri and Pal and others which can be derived easily, for example, $k.k' = p.k - p'.k$ [in turn, using $p - p' = k' - k$ and dotting k on both sides, with $k^2 = 0$.]

Finally, (as usual) be careful with issues such as (i) combinatorial factors due to identical particles (see discussion in Lahiri and Pal above Eq. 13.64: this comment applies to HW 2.3 as well), (ii) signs (especially of momenta in zeta vertices – again, this comment applies to HW 3.3 as well, but only *if* you use radial representation for it) and (iii) existence of multiple diagrams (this comment applies to HW 2.3 as well).

4 Homework 4 [Spontaneous (Gauge) Symmetry Breaking: Higgs mechanism], due Monday, March 28 on ELMS

4.1 Propagators in R_ξ gauge

Exercise 13.16 of Lahiri and Pal.

Again, derive these propagators along the lines of section 8.2, i.e., as a (classical) Green's function.

4.2 Nambu-Goldstone boson equivalence theorem

We mentioned in lecture that in the Higgs mechanism, the gauge boson becomes massive by "eating" the "would-be" Nambu-Goldstone boson (i.e., the massless scalar of the theory *before* gauging of the – spontaneously broken – global symmetry), i.e., the extra/longitudinal degree of freedom polarization of a massive gauge boson "corresponds" to this Nambu-Goldstone boson.

In fact, it turns out that this statement can be made more precise in the form of the Nambu-Goldstone boson equivalence theorem as follows: the amplitude for emission/absorption of a longitudinally polarized gauge boson becomes equal, at energies much larger than its mass, to the amplitude for emission/absorption of the Nambu-Goldstone boson that was eaten.

The goal of this problem is to illustrate this theorem: see, for example, Peskin and Schroeder's section 21.1 for further discussion.

Consider the theory of an "electron" (denoted by ψ) and complex/charged scalar (denoted by ϕ) coupled to the "photon"[2]:

$$\mathcal{L} = \bar{\psi}\left(i\gamma^{\mu}D_{\mu} - m\right)\psi + \left(D^{\mu}\phi\right)^{\dagger}\left(D_{\mu}\phi\right) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{7}$$

Suppose we include the following potential for the scalar (with $\mu^2 < 0$)

$$V\left(\phi\right) = \mu^2\phi^{\dagger}\phi + \lambda\left(\phi^{\dagger}\phi\right)^2 \tag{8}$$

which results in scalar field acquiring a non-zero VEV and the photon a mass, M_A (as usual).

As in lecture, the physical (massive) scalar which is "left-over" is denoted by η (and called the "Higgs" boson) and the (would-be) Nambu-Goldstone boson by ζ.

Note: at the renormalizable level, there is *no* gauge-invariant (direct) coupling of electron to scalar.

4.2.1 e^+e^- annihilation into longitudinal photons

(i) In *unitary* gauge, calculate the Feynman *amplitude* (only, i.e., *no* need to calculate the cross-section) for electron-positron annihilation into *longitudinal* polarizations of the (massive) photon:

$$e^+e^- \rightarrow \gamma_L\gamma_L, \tag{9}$$

[2] Note that the coupling of electron to photon here is "vector-like", i.e., both chiralities couple identically. A similar discussion of the equivalence theorem for case with, say, only left-chirality of the electron coupled to the photon is left to you as an "informal" homework.

where the subscript "L" on γ denotes longitudinal polarization, i.e., use the following polarization vectors for the final state photons (with 3-momentum $\mathbf{k}$ and energy E):

$$\begin{aligned} \epsilon^{\mu}_{L}(k) &= \frac{1}{M_A}\left(|\mathbf{k}|, \mathbf{E}\hat{\mathbf{k}}\right), \quad \text{where } \hat{\mathbf{k}} \text{ is a unit vector along } \mathbf{k} \\ &\approx \frac{k^{\mu}}{M_A}, \quad \text{for } E \gg M_A. \end{aligned} \tag{10}$$

Work in the limit of center-of-mass energy ($\sqrt{s}$) being much larger than the photon (and electron) mass.

Consider *tree*-level diagrams only.

(ii) Next, calculate the amplitude for electron-positron annihilation into two Nambu-Goldstone bosons (say, in R_ξ gauge[3] with *finite* ξ)

$$e^+e^- \;\; \rightarrow \;\; \zeta\zeta \tag{11}$$

again in the high-energy limit (i.e., neglect masses of photon and the spin-0 particles compared to the center-of-mass energy) and at *tree*-level.

Note: the Feynman rules for coupling of Nambu-Goldstone bosons (and Higgs boson) to the photon can be obtained from the ones for scalar QED given in problem 9.1 (a) on page 312 in Peskin and Schroeder or page 505 of appendix in Cheng and Li or page 1 of notes from Phys624 that I taught in Fall 2010 at
http://www.physics.umd.edu/courses/Phys624/agashe/F10/notes/scalarQED.pdf.

You should find that the the above two amplitudes (even if vanishing!) are the same.

4.2.2 e^+e^- annihilation into longitudinal photon and Higgs boson

Repeat the above steps for this process (again, in the high-energy limit), i.e., calculate the amplitude for

$$e^+e^- \;\; \rightarrow \;\; \gamma_L \eta, \tag{12}$$

Again, consider *tree*-level diagrams only.

Note: since electron current to which the photon couples is conserved, will the $k_\mu k_\nu$ part of photon propagator (where k is its momentum) contribute?

Then, compare it to the amplitude for

$$e^+e^- \;\; \rightarrow \;\; \zeta\eta. \tag{13}$$

You should again find that the two amplitudes are the same.

[3] Of course, in general, the Feynman rules are that would-be Nambu-Goldstone bosons should only appear as *internal* lines, but here it is OK to have them as final states *just* for the purpose of illustrating the Nambu-Goldstone boson equivalence theorem.

5 Homework 5 (Non-abelian *global* symmetry), due Monday, April 4 on ELMS

5.1 Invariance of the Pion-Nucleon Lagrangian

Exercise 13.7 of Lahiri and Pal.

[In the first part of this problem, the *non*-trivial part is mostly showing that the nulceon-pion interaction is invariant under the (different) transformation on pion and nucleon fields.]

[In the second part, think about whether (or not) the equations of motion same as the case of free fields.]

5.2 Adjoint representation

5.2.1 Structure constants

Exercise 13.10 of Lahiri and Pal.

[In lecture, we already "argued" that matrices defined in terms of structure constants of the $SU(2)$ group form a (adjoint) representation of the same group. To be more precise, I pointed out that the 3×3 matrices in Eq. 13.16 of Lahiri and Pal "look like" ϵ_{abc}. And, I asked you to check that the commutation relations of these matrices is indeed same as that of the 2×2 Pauli matrices. So, the goal here is to show this is true for $SU(n)$ groups in general.]

5.2.2 $n \times n$ matrix

Consider a set of fields, ϕ_a, where $a = 1$ to $(n^2 - 1)$, which transform as an adjoint representation of the $SU(n)$ group, i.e., transforming as

$$\phi'_a \;=\; \exp\left(-i\beta_b \left[T^b_{(\text{adj})}\right]_{ac}\right)\phi_c \tag{14}$$

where $T^b_{(\text{adj})}$ denotes the generators in adjoint representation and β_b are the parameters of this transformation.

It *can* be shown that the $n \times n$ matrix defined by

$$\Phi \;=\; \phi_a T^a_{\text{fund}}, \tag{15}$$

where T^a_{fund} are the generators of the fundamental representation, transforms as

$$\Phi' \;=\; \exp\left(-i\beta_a T^a_{\text{fund}}\right)\Phi\exp\left(+i\beta_b T^b_{\text{fund}}\right) \tag{16}$$

i.e., as a fundamental representation from the left-hand side and *anti*-fundamental one from the right-hand side.

For this homework problem, show (only) that the *infinitesimal* version of the relation in Eq. (16) is true.

[We will uses the above relation in deducing that the gauge fields transform as adjoint representation.]

5.3 Spontaneous breaking of $SU(2)$ with triplet VEV

This problem is based on the spontaneous symmetry breaking of section 13.4.3 of Lahiri and Pal which was also discussed in lecture.

5.3.1 Remnant symmetry: part I

Exercise 13.11 of Lahiri and Pal. Of course as part of the above problem, you should also find that the (fluctuation around VEV of) scalar ϕ_3 is massive, but $\phi_{1,2}$ are massless.

The idea here is mostly to show the remnant symmetry using "brute force", i.e., simply *plug* the VEV in Eq. 13.49 of Lahiri and Pal in Lagrangian of Eq. 13.47 and then "chug" along. In lecture, we already mentioned the above "expectation" of remnant symmetry (based on a simpler/perhaps more elegant argument).

[The rank of a $SU(n)$ group can be defined as the number of diagonal generators in the fundamental representation. So, the rank is 1 for $SU(2)$ and $U(1)$ groups. Thus, we find in this $SU(2)$ example that the rank is unchanged when a (scalar) field in adjoint representation acquires a VEV – it turns out that this statement is true for general $SU(n)$ groups as well.]

5.3.2 Remnant symmetry: part II

Another elegant way to figure out if there is a remnant symmetry (in general) is as follows. Suppose a generator "annihilates" the vacuum, i.e.,

$$T_a \phi_0 \;=\; 0 \tag{17}$$

where ϕ_0 denotes the (constant) field configuration corresponding to the minimum of the potential. Then the corresponding transformation leaves the vacuum invariant, i.e.,

$$\exp\left(-i\beta_a T_a\right)\phi_0 \;=\; \phi_0 \tag{18}$$

and thus gives an unbroken symmetry. Conversely, a generator which *does* annihilate the vacuum corresponds to a broken symmetry.

(i) For the above (specific) model, is there any generator, i.e., one of the three T's in Eq. 13.16, which satisfies Eq. 17 above?

(ii) Show that the transformation corresponding to the generator in part (i) above is precisely the symmetry you found in part 5.3.1 above (by brute force).

[All of the above ways do *not* make use of/refer to the Goldstone's theorem – that is the subject of next problem.]

5.3.3 "Identification" of Nambu-Nambu-Goldstone bosons

In lecture, we discussed the counting of number of massless scalars based on Goldstone's theorem for this model, i.e., thinking of these scalars as Nambu-Nambu-Goldstone bosons. However, we can do a further check using Goldstone's theorem as follows.

Based on part 5.3.2 above, we know the generators (and thus the transformations) which correspond to the (spontaneously) broken symmetries. So, we can determine the identity of the corresponding Nambu-Goldstone bosons for this model in the form of the "eigenvectors" in field space which were derived in the context of the proof of Goldstone's theorem discussed in class (which is, in turn, based on Peskin and Schroeder, pages 351-352).

Show that the scalars $\phi_{1,2}$ – which you found to be massless in part 5.3.1 above – correspond precisely to the above eigenvectors.

[Again, if the transformations of the fields are given by

$$\phi^a \to \phi^a + \alpha \Delta^a(\phi) \tag{19}$$

(where α is a small parameter), then the Nambu-Goldstone bosons are the vectors (in field space) given by $\Delta^a(\phi_0)$, where ϕ_0 is the VEV configuration in field space.

Just to be clear, we already showed in lecture that the identity of Nambu-Goldstone bosons is as above for the *general* case (so there is no need to repeat that proof here). Rather, the idea is to verify explicitly that this identity applies for the above *particular* example.

In words, the above formula says that we should look for the combination of fields which transforms *in*homogeneously (i.e., *not* just into other fields) *when evaluated in the ground state.* For example, for the simplest case (discussed in section 13.4.2 of Lahiri and Pal and in lecture) of a $U(1)$ symmetry, i.e., VEV for a 1 complex scalar (i.e., for one of two real scalars), we have the (infinitesimal) transformation:

$$\begin{aligned} \eta &\to \eta - \alpha\zeta \text{ (i.e., homogeneous) vs.} \\ \zeta &\to \zeta + \alpha(v+\eta) \text{ (where } \alpha v \text{ is the } \textit{in}\text{homogeneous part)} \end{aligned} \tag{20}$$

so that indeed ζ is massless.]

5.4 Spontaneous breaking of $SU(2)$ with doublet VEV

Consider a doublet of *complex* scalar fields,

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{21}$$

where ϕ_1 and ϕ_2 are complex scalar fields, with the following $SU(2)$-invariant Lagrangian

$$\mathcal{L}_\phi = (\partial^\mu \Phi)^\dagger (\partial_\mu \Phi) - \mu^2 \Phi^\dagger \Phi - \lambda \left(\Phi^\dagger \Phi\right)^2, \tag{22}$$

where

$$\Phi^\dagger \Phi = \phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2 \tag{23}$$

5.4.1 Spectrum

If this multiplet acquires a VEV, show that three mass*less* scalars ("likely" Nambu-Goldstone bosons) emerge and that there is one massive real scalar field leftover.

[It might be useful here to work with a radial representation for complex scalar fields, just like we did in $U(1)$ case in lecture and in Lahiri and Pal. However, remember that we have a *doublet* of such fields here so that the radial representation that we used for *single* complex scalar field might have to be suitably modified.

In case you really (!) need help with this radial approach for $SU(2)$, then you can take a look at Eq. 11.64 on page 349 of Cheng and Li.]

5.4.2 Remnant symmetry

Determine if the resulting Lagrangian (including interactions for these Nambu-Goldstone bosons and the massive real scalar) has a *remnant* symmetry.

Hint: Note that the idea here is to keep using *complex* doublet representation for the scalar fields. However, it is actually easier to see if there is a remnant symmetry in a different representation as follows (but again *try* not use the latter one since the goal is to show equivalence of different representations, even if involves hard work, just like the case of radial vs. linear representation that was discussed in homework 3.4). Namely, suppose we re-write Φ in terms of four *real* scalar fields, denoted by $\tilde{\phi}_i$ ($i = 1$ to 4). It is clear that the potential above depends only on $\sum_{i=1}^{i=4} \tilde{\phi}_i^2$, i.e., the "length" of Φ thought of as a (real) "four-vector". Thus the Lagrangian actually has a larger-than-$SU(2)$ symmetry, namely, $SO(4)$ [which is *locally* same as $SU(2) \times SU(2)$]. The VEV of one of these $\tilde{\phi}_i$ clearly breaks this down to a $SO(3)$ [$\approx SU(2)$] symmetry.

[This is a rough picture of the spontaneous symmetry breaking of the electroweak (EW) *gauge* symmetry in the Standard Model (SM). In particular, we will *return* to this (initial) larger-than-$SU(2)$ symmetry in the context of HW 7.1.2 (where we consider a gauged version of above model) and (relatedly) during discussion in lecture of the theory of the EW sector of the SM, i.e., ratio of W and Z gauge boson masses, arising from breaking of the *full* – $SU(2) \times U(1)_Y$ – EW gauge symmetry. Whereas, HW 8.4 concerns a *different* model for the breaking of this SM $SU(2) \times U(1)_Y$ gauge symmetry, which does not incorporate such an enhanced symmetry, hence ratio of W and Z masses will be modified relative to the SM prediction with a Higgs (scalar) doublet instead.]

[Also, we see in this example that the rank of the group *is* reduced when a scalar field in the *fundamental* representation acquires a VEV (cf. adjoint case discussed above) – again, it turns out that this statement is true for general $SU(n)$ groups as well.]

6 Homework 6 [Non-abelian *gauge* symmetry (classical and unbroken)], due Monday, April 11 on ELMS

6.1 Transformation of covariant derivative of fermion

Exercise 14.1 of Lahiri and Pal.

[The idea here is to show that the transformation of the *non*-abelian *co*variant (but not of ordinary) derivative acting on fermion does *not* involve derivative of transformation parameters, just like for abelian case which is discussed in Exercise 9.2; in fact, this covariant derivative transforms exactly like the fermion itself, i.e., the covariant derivative part "goes along for the ride" in this sense. One can think of this as a further check that the transformation in Eq. 14.8 of Lahiri and Pal "does the job".]

6.2 Gauging symmetry for scalar

Exercise 14.2 of Lahiri and Pal.

[In lecture (and in Lahiri and Pal), we deduced the necessity of introducing the gauge fields (and their transformation properties) based on invariance under *local* $SU(n)$ transformations for fermions in fundamental representation. So, the idea here is to repeat it for *scalar* field(s) in fundamental representation.]

6.3 Covariant derivative of *gauge* fields

6.3.1 *Transformation* of covariant derivative of gauge fields under (gauge) transformation of gauge field

Exercise 14.3 of Lahiri and Pal.

[You can simply start with Eq. 14.21 of Lahiri and Pal for this purpose. However, as informal homework, you should try to work out the algebra before (which was not done in lecture). The idea here is to show that the transformation of the covariant derivative of gauge fields does *not* involve derivative of transformation parameters, just like for fermion case which is discussed in above problem; in fact, this covariant derivative transforms locally exactly like the gauge field itself, but restricting to *global* transformation.]

6.3.2 *Alternative* way to deduce expression for covariant derivative of gauge fields

The idea here is to come up with another way (cf. one discussed in section 14.2 of Lahiri and Pal and touched up on in lecture) to deduce the (same) expression for covariant derivative of non-abelian gauge fields, based (again) on generalizing the abelian case.

Abelian case

(i) Show that the (usual) antisymmetric/field strength tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{24}$$

is related to the covariant derivative of fermion fields as follows:

$$(D_\mu D_\nu - D_\nu D_\mu)\,\psi \;\; = \;\; ieQF_{\mu\nu}\psi \tag{25}$$

(ii) Based on the above relation *and* the gauge transformation of the covariant derivative of fermion, determine the transformation property of the field strength tensor, i.e., (re-)derive its gauge invariance property.

Generalizing to *non*-abelian case

In analogy with the abelian case, *define* the (candidate) field strength tensor, $F^a_{\mu\nu}$, by

$$ig\left(T_{a\,\text{fund}}F^a_{\mu\nu}\right)\Psi \;\; \equiv \;\; (D_\mu D_\nu - D_\nu D_\mu)\,\Psi \tag{26}$$

where Ψ is a fermion field(s) transforming as a fundamental representation of $SU(n)$ symmetry group, i.e.,

$$\Psi' \;\; = \;\; U\Psi \tag{27}$$

with

$$U \;\; = \;\; \exp\Big[-ig\beta_a(x)T^a\Big] \tag{28}$$

where (as usual) β_a are the parameters of the gauge transformation and T^a_{fund} are the generators of fundamental representation of $SU(n)$.

(iii) Show that (as already indicated at the beginning of this problem) the above definition of field strength tensor is actually *identical* to the one in Eq. 14.20 (and 14.21) of Lahiri and Pal (and discussed in lecture), i.e.,

$$T_{a\,\text{fund}}F^a_{\mu\nu} \;\; = \;\; \partial_\mu\left(T_aA^a_\nu\right) - \partial_\nu\left(T_aA^a_\mu\right) + ig\Big[T_bA^b_\mu, T_cA^c_\nu\Big]_- \tag{29}$$

(iv) Using the gauge transformation property of the covariant derivative of fermion *and* Eq. 26, show that the field strength tensor (defined above in terms of the covariant derivative of fermion) transforms as

$$T_{a\,\text{fund}}F^{a\,\prime}_{\mu\nu} \;\; = \;\; U\left(T_{a\,\text{fund}}F^a_{\mu\nu}\right)U^{-1}, \tag{30}$$

i.e., same as the transformation of the field strength tensor given in Eq. 14.22 of Lahiri and Pal (and worked out in problem 6.3.1 above using a different approach).

6.3.3 Analog of *in*homogeneous Maxwell equations

Exercise 14.9 of Lahiri and Pal.

[The idea here is to show that the *non*-abelian gauge fields also satisfy analogs of the inhomogeneous Maxwell equations (with the currents made only of *non*-gauge fields), *provided* we replace the ordinary derivatives by *co*variant ones.]

Note: there seem to be some "typos" in the expression for conserved current in Eq. 14.36 of Lahiri and Pal (which you will discover as you work out the problem):

(i). One of Φ's should have a "dagger" (i.e., Hermitian conjugate of Φ).

(ii). The ordinary derivative on Φ (i.e., ∂^μ) should be replaced by covariant (i.e., D^μ), i.e., the scalar field contribution to the current (*bi*-linear in scalar field of course) actually has a term containing also a *gauge* field (obviously, this point applies to *abelian* gauge theory as well). Thus, it seems that the current is modified when we go from global symmetry to gauge symmetry. Indeed, this issue stems from gauge interaction involving *derivative* of scalar field, thus it is related to the "difficulty" in canonical quantization discussed during lecture, where (similar to current) the conjugate momentum, Π_Φ is - at least naively – shifted due to this interaction.

Having said this, you should (double-)check that this term (again, with two scalar and one gauge field) in current does not cancel when we add "h.c.").

Also, compare this situation to the fermion (Ψ) case, where gauged and global theory currents (and conjugate momenta) are the same (since there is no derivative in gauge interaction).

(iii). You could put an overall factor of gauge coupling (g) either in the expression for current in Eq. 14.36 *or* on RHS of the EOM in Eqs. 14.37, 14.38.

7 Homework 7 (Spontaneous breaking and quantization of non-abelian gauge symmetry), due Monday, April 18 on ELMS

7.1 Higgs mechanism for $SU(2)$

The idea here it to "repeat" the spontaneous symmetry breakings studied in problems 5.3 and 5.4, but now with *gauged* symmetries (instead of global).

7.1.1 Adjoint VEV

(i) Suppose we gauge the global $SU(2)$ symmetry of problem 5.3 , i.e., with VEV for $SU(2)$ triplet (real) scalar field. Calculate the spectrum of gauge boson masses.

(ii) *If* there is a mass*less* gauge boson after the spontaneous symmetry breaking, then show that its "identity" (in the space of the original, i.e., massless gauge bosons) agrees with the general theorem discussed in lecture (in turn, based on pages 692-693 of Peskin and Schroeder).

7.1.2 Doublet VEV

(i) Similarly, calculate the spectrum of gauge bosons for the case of gauging the $SU(2)$ global symmetry of problem 5.4, i.e., where a (complex) scalar doublet under this symmetry acquires VEV.

(ii) If you find a relation between masses of (massive) gauge bosons, then show that it was expected based on the enhanced symmetry that we start with (and thus the remnant symmetry after spontaneous symmetry breaking) discussed in problem 5.4.

[As mentioned in the context of HW 5.4 (which was the *global* version of the above model), this is a "simplified" picture of the breaking of the electroweak (EW) gauge symmetry of the Standard Model (SM). We will return to this point during the lecture on the theory of the EW sector of the SM, i.e., ratio of W and Z masses, arising from breaking of the full – $SU(2) \times U(1)_Y$ – EW gauge symmetry by the Higgs (scalar) doublet. Also, see HW 8.4 for prediction of the ratio of W and Z masses with the breaking of this gauge symmetry achieved by a scalar field transforming as a *different* representation of $SU(2) \times U(1)_Y$.]

7.2 Feynman rules for scalar-gauge boson vertex

Exercise 14.7 of Lahiri and Pal

In lecture, we derived the Feynman rule for the vertex with fermion and gauge boson. Here, the idea is to do it for scalar-gauge boson vertex. Again, derivatives will get converted into momenta, but be careful with signs!

7.3 Feynman rules for quartic gauge boson vertex

Exercise 14.8 of Lahiri and Pal

In lecture, we derived (at least part of) the Feynman rule for trilinear gauge boson vertex. Here, the idea is to do it for quartic vertex.

Note: there's a typo in Lahiri and Pal's Fig. 14.1 in the Feynman rule for the quartic coupling: see
http://www.saha.ac.in/theory/palashbaran.pal/books/qft/2ed/err2.html

8 Homework 8 (Theory of Standard Model), due Monday, May 2 on ELMS

8.1 Another quartic interaction term for Higgs doublet?

As mentioned in lecture, the goal of this problem is to figure out if there is a quartic interaction term for the Higgs doublet, in addition to the $\left(\Phi^\dagger \Phi\right)^2$ term that we have included so far.

8.1.1 Product of adjoints

Suppose there is a field, ϕ_a ($a = 1...n^2 - 1$)[4], which is in adjoint representation of $SU(n)$. Show that "product" of two such (possibly different) fields, i.e., $\phi^a \phi'_a$ (repeated indices – here a – are summed over, unless explicitly stated otherwise)[5] is invariant under (local) $SU(n)$ transformations.

Restrict yourself to the *infinitesimal* version of this transformation only, although the result can be shown to hold for finite case as well.

8.1.2 Building an adjoint from general representation

Show that we can construct a ("effectively") real scalar field in adjoint representation by (appropriately) "multiplying" a field in any representation (whether scalar or fermion) and its conjugate. Namely, consider the object

$$\phi^a \equiv F_i^* T^a_{ij} F_j, \tag{31}$$

where F_i are components in $SU(n)$ space of a field in *any* representation, F^* denotes hermitian conjugate of field[6] and T^a are the generators in the corresponding representation.

(i) Show that ϕ^a above is "real", i.e., it is its own hermitian conjugate.

(ii) Show that ϕ^a above transforms in adjoint representation.

Again, it is enough to do it for infinitesimal transformations *only* (although the result can be shown to valid in general).

8.1.3 Fermion coupling to gauge boson

Based on above result (simply *assume* its validity for full version of the transformation),

(i) show that the term corresponding to the coupling of fermionic matter field in any representation to the gauge boson, i.e., $\left(\bar{\Psi} T^a \Psi\right) A^\mu_a$, is invariant under *global* $SU(n)$ transformations. Here, you are supposed to make use of the known way in which A^a_μ transforms under *global* transformation.

Of course the invariance of the above term is much more clear when we instead consider the transformation of the "combination" $(T^a A^\mu_a)$ as in Eq. 14.8 and that of Ψ as in Eq. 14.2 of Lahiri and Pal.

(ii) Is the above term invariant under *local* $SU(n)$ transformations?

(iii) If the answer to previous question is "No", then which other term in Lagrangian cancels the shift of above term?

[4]If you wish (for example for intuition or a better feel for this problem), you can imagine this to be a scalar field, even though it can be a fermion (for example). Even if it is a scalar field, note that it can be either real or complex.

[5]Note that lower and upper index of internal symmetry have the same meaning, unlike for Lorentz index. So, here lower and upper indices are used just for purpose of presentation.

[6]In order not to confuse with operations in $SU(n)$ space, I am not using "dagger" here.

8.1.4 Another invariant formed out of Higgs doublet?

(i) Finally, suppose we have a complex doublet of $SU(2)$, as in Eq. (21). Using above results, is the term $\left(\Phi^{\dagger}\sigma_a\Phi\right)\left(\Phi^{\dagger}\sigma^a\Phi\right)$ invariant under $SU(2)$ transformations? [Note that $\left(\Phi^{\dagger}\sigma_a\Phi\right)$ is a "number" in $SU(2)$ *doublet* space.]

(ii) Is the above term its own hermitian conjugate (i.e., can it be added to the Lagrangian by itself)?

(iii) Using Pauli-matrix properties, show that the above (new) term is identical to the quartic term which we had already considered (i.e., included in the Lagrangian)?

(iii) In the full standard model, the Higgs doublet also has non-zero hypercharge. Then, is the above combination also invariant under hypercharge?

8.2 Unitary gauge

Exercise 15.3 of Lahiri and Pal

As discussed in lecture, it is convenient to use a *radial* representation for the Higgs doublet in order to go to unitary gauge (which contains only the physical degrees of freedom).

We already did this for (pure) $U(1)$ gauge theory in lecture and you might have done it for $SU(2)$ gauge theory in homework 5.4. Here, the idea is to do it for $SU(2) \times U(1)$.

8.3 "New" Higgs doublet

Exercise 15.8 of Lahiri and Pal.

As discussed in lecture, for the above reason we must use $\tilde{\phi}$ to give masses to up-type quarks in the Standard Model (vs. using ϕ for down-type quarks and charged leptons) and in supersymmetric extension, we actually require *two* (independent) Higgs doublets.

8.4 ρ parameter with triplet (and doublet)

During the lecture on the theory of electroweak (EW) sector of the SM, we discussed how the breaking of $SU(2) \times U(1)_Y$ gauge symmetry by the Higgs (scalar) doublet leads to a specific prediction for the ratio of W and Z gauge boson masses, which is in excellent agreement with the experimental measurement. In turn, this feature arises because the Higgs doublet model has an enhanced symmetry to begin with as originally discussed in HW 5.4 in the global symmetry context (and in HW 7.1.2 for a "toy" version of the SM EW gauged symmetry). Here, we will investigate whether prediction for such a ratio of W and Z masses is maintained if they arise from VEV of a suitable (scalar) triplet instead.

Consider a (complex) scalar, which is a triplet of $SU(2)$ and with hypercharge $+1$ and hence is denoted by $\mathcal{T}_1$. Suppose we *add* $\mathcal{T}_1$ (and its hermitian conjugate) to the Standard Model, i.e., we still have the (complex) scalar doublet, ϕ as usual.[7]

[7]VEV of this new scalar field can be used to give neutrinos a mass, with*out* adding a right-handed

Assume that $\mathcal{T}_1$ gets a VEV which is denoted by v_T (up to a normalization factor which is not really relevant here). We would like to not break electric charge (Q) in this process so that this VEV must be in zero electric charge component of $\mathcal{T}_1$. Since $\mathcal{T}_1$ has hypercharge, $Y = 1$, it is clear that this VEV will have to be in the "direction" (in $\mathcal{T}_1$-space) such that eigenvalue of T_3 is -1 [as usual, T_3 denotes the "third" $SU(2)$ generator, i.e., it is the third component of isospin].[8]

Hint: Is the basis in Eq. 13.16 of Lahiri and Pal that of eigenstates of T_3 [i.e., does $(0,0,1)^T$ have eigenvalue -1]? You can either keep using this basis or "switch" to a different one, for example, see example 2 on page 93 of Cheng and Li.

(i) Just like the VEV of ϕ (denoted by v_ϕ), v_T contributes to W and Z masses (but obviously not to photon mass). Show that the new theory prediction is

$$\begin{aligned}\rho_{\rm th} &\equiv \frac{M_W^2}{M_Z^2 \cos^2\theta_W} \\ &= \frac{v_\phi^2 + 2v_T^2}{v_\phi^2 + 4v_T^2}\end{aligned} \tag{32}$$

[It might be useful to re-write $\mathcal{T}_1$ as a 2×2 matrix (by multiplying the triplet by Pauli matrices), although you don't have to do it this way].

(ii) Assume that the experimental measurement is

$$\rho_{\rm expt} = 1 \pm 0.001 \tag{33}$$

What is the constraint then on the size of triplet VEV relative to the doublet VEV?

8.5 $SU(2)$ model of Georgi and Glashow, Physical Review Letters 28, 1494 (1972)

(This problem is a bit unusual/difficult, but it gives you a taste of "model-building")

As mentioned in lecture, it is possible to identify the neutral W boson (W_3) of $SU(2)$ with the photon provided the fermions are *not* in a doublet representation. Here is an example of such a theory (focussing only on the lepton sector) – of course, it predicts that there are no other neutral currents and so was eventually ruled out by the latter discovery.

(i) To begin with, show that we can*not* use (complex) scalar doublet to break the electroweak gauge symmetry (of course, in the *desired* pattern, i.e., keeping photon massless) in this case (unlike in the Standard Model).

(ii) However, a *real* scalar $SU(2)$ triplet, denoted by $\mathcal{T}$, is suitable for this purpose (see also homework problems 5.3 and 7.1.1). Suppose we identify $Q = T_3$ [again, T_3 is 3rd generator

neutrino.

[8] Such a orientation of $\mathcal{T}_1$ VEV can be arranged by suitable couplings of $\mathcal{T}$ to ϕ.

of $SU(2)$]. What is the T_3-eigenvalue of the direction (in $\mathcal{T}_1$-space) which should obtain a VEV?

Pick a basis for representing $\mathcal{T}$. In this basis, which component gets a VEV?

The fermion sector of the model consists of the following two $SU(2)$ triplets (in the appropriate *basis*):

$$
\begin{aligned}
\Psi_R &\equiv \begin{pmatrix} E^+ \\ N^0 \\ e^- \end{pmatrix}_R \quad \text{and} \\
\Psi_L &\equiv \begin{pmatrix} E^+ \\ N'^{\,0} \\ e^- \end{pmatrix}_L
\end{aligned}
\tag{34}
$$

and the $SU(2)$ singlet

$$
\nu_L'^{\,0} \tag{35}
$$

(iii) Show that the electric charges of the various fermions (which are shown as superscript above) are indeed correct (again, with a *suitable* choice of basis).

We can include the following gauge [i.e, $SU(2)$]-invariant bare mass term

$$
\mathcal{L} \ni M\overline{\Psi_L}\Psi_R, \tag{36}
$$

with $SU(2)$ indices suitable contracted.

And, the following Yukawa couplings of scalar triplet ($\mathcal{T}$) to fermions

$$
\mathcal{L} \ni i\, h_e \mathcal{T}\overline{\Psi_L}\Psi_R + h_N \overline{\nu_L'}\Psi_R \mathcal{T} \tag{37}
$$

[where $h_{e,\,N}$ are (purely) *real* and note the "i" in front of h_e, i.e., the 1st coupling is (purely) *imaginary*] again with suitable contraction of $SU(2)$ indices so as to form $SU(2)$-invariants.

Note that this operation for 1st term in Eq. (37) is different than for the term in Eq. (36) (even though $\overline{\Psi_L}\Psi_R$ appears in both places) – you will need to figure out how to do this. See, for example, homework problems 8.1.1 and 8.1.2. Equivalently, since Ψ is a $SU(2)$ triplet, $\bar{\Psi}$, i.e., Ψ^*, is an *anti*-triplet. However, we can re-write it as triplet (just like was done in lecture for doublet): see discussion around Eq. 4.72 of Cheng and Li. Then, just think of an $SU(2)$ [i.e., "isospin"] triplet as an actual spin/angular momentum 1 object and 1st term in Eq. (37) as product of 3 such objects. As check that you have an $SU(2)$ invariant, consider adding T_3 charges of all fields in each term in the Lagrangian – it should give 0.

[Just to repeat the "alert", in all of the above and below calculations, you have to be(very) *careful* with choices of *bases* for $\mathcal{T}$ *and* for $\Psi_{L,\,R}$: see HW 8.4 above also. Note that the answers are mostly given, so you should be able to use that as a "cross-check" for your group theory algebra.]

These terms will give additional contribution to fermion masses when scalar acquires VEV.

(iv) First, consider the effects of the above terms in the neutral fermion sector. Find the mixing angle (β) – in terms of above M, h_N and v_T (VEV of triplet, up to normalization factor, which is not really releant here) so that the combintation

$$\nu_L \equiv \sin\beta N_L' - \cos\beta \nu_L' \tag{38}$$

remains massless and thus can be identified with usual neutrino.

(v) The orthogonal combination of ν_L' and N_L' pairs with N_R to become heavy, with mass denoted by M_N. Show that

$$M_N = \frac{M}{\cos\beta} \tag{39}$$

(vi) In charged fermion sector, show that the resulting masses are

$$\begin{aligned} m_e &= M - h_e v_T \\ M_E &= M + h_e v_T \end{aligned} \tag{40}$$

Thus, we can choose the coupling h_e so that E^+ is very heavy, but e^- remains light so that the latter can then be identified with usual electron.

(vii) Working out the gauge interactions from covariant derivatives (as usual), but rewriting them in terms of mass eigenstates, show that the W boson and A (photon) couplings are given by

$$\mathcal{L} \ni +\frac{1}{2}e \sin\beta\, W_\mu \bar{e}\gamma^\mu (1-\gamma_5)\,\nu + \text{h.c.} - e\, A_\mu \overline{e^-}\gamma^\mu e^- ... \tag{41}$$

where the "..." denote terms with the heavy N and E fields. Thus, in the limit of heavy N and E, this theory reduces to the usual one.

(viii) Based on the experimental values of G_F (the coefficient of effective 4-fermion operator obtained from W boson exchange) and e, show that in this model

$$M_W \leq 53 \text{ GeV} \tag{42}$$

Of course, the discovery of W gauge boson in early 1980's with mass of ≈ 80 GeV also rules out this model.

9 Homework 9 [Phenomenology of Standard Model (I)], due Monday, May 9 on ELMS

9.1 ν_e e scattering

Exercise 15.15 of Lahiri and Pal.

This problem is somewhat tedious (most likely, last one of this kind for this course!). You should feel free to use software programs for doing some of the algebra. If required, you can

go back to section 7.5 of Lahiri and Pal or notes from Phys 624 that I taught in Fall semester of 2010 at
http://www.physics.umd.edu/courses/Phys624/agashe/F10/notes/cross-section.pdf, where cross-section for a similar process was calculated.

Note: are the $g_{V,\,A}$ shown in Eq. 15.93 of Lahiri and Pal (which determine the cross-section for the process in this exercise, i.e., $\nu_e e^- \rightarrow \nu_e e^-$) same as what should be used in Eqs. 15.89, 15.92 for the *other* process already worked out earlier in this section (i.e., $\nu_\mu e^- \rightarrow \nu_\mu e^-$)? (In case you need a reminder, the values of $g_{V,\,A}$ entering $\nu_\mu e^- \rightarrow \nu_\mu e^-$ are given in Eq. 15.73.)

As discussed in lecture (and hinted at in section 15.5.2 of Lahiri and Pal), the elastic scattering of *muon*-neutrinos off of electrons (calculated in the main body of this section) fixes the couplings of electron/muon to Z boson only up to *discrete ambiguities*. So, we need *more* (independent) measurement(s) of these couplings such as is provided (see note above) by scattering of *electron*-neutrinos off of electrons which is *this* homework problem/Lahiri and Pal exercise (in turn, this is due to the extra contribution from W boson exchange in the latter process).

As also discussed in lecture, the *existence* of neutral *weak* currents was *first* established (hence confirming the Standard Model/EW theory prediction, circa 1970's) by similar reactions, but with neutrinos scattering off of *hadrons* instead of electron.

9.2 Forward-backward asymmetry in e^+e^- annihilation into $\mu^+\mu^-$

Exercise 15.13 of Lahiri and Pal.

Again (as discussed in lecture and in above homework problem), it is useful to have *several* (independent) neutral current measurements in order to (completely/uniquely: again, including *signs*) determine the couplings of electron/muon to Z boson, hence the parameters g and g' of the EW theory *separately* (using also QED fine structure constant). The above processes (i.e., taken together) do this job: see section 12.1 on page 364 onwards, especially Fig. 12.3 on page 367, of Cheng and Li for more details.

All the above experiments thus played an important role in the establishment of the electroweak theory, even before the W and Z bosons were actually discovered (in early 1980's). Nowadays, they continue to provide (more stringent than in 1970's) tests of the Standard Model (and any proposed extensions of it since the latter will, in general, modify the predictions for these processes).

9.3 Decay of Z boson

Exercise 15.10 of Lahiri and Pal

For better accuracy, use the value of QED α evaluated at energy M_Z.

The experimental measurement of the total "invisible" decay width of Z (done in 1990's) is in agreement with $3\times$ the above prediction, which can be thought of as evidence for the existence of three (and *not* more)[9] neutrinos (since neutrinos would not be detected in these experiments and thus count as invisible).

9.4 Non-abelian electroweak gauge vertex

Exercise 15.16 of Lahiri and Pal.

(The above couplings are used in the calculation of $e^+e^- \rightarrow W^+W^-$: this derivation was outlined during lecture, so idea here is for you to repeat/finish it.)

9.5 Equivalence theorem

Exercise 15.17 of Lahiri and Pal.

We discussed *equivalence theorem* in lecture and in homework 3.2 in the context of Higgs mechanism for an abelian gauge theory (*before* coming to the Standard Model). Now, we see an application of this theorem in the context of the Standard Model.

For this calculation, use – in the s-channel – (i) both photon and Z boson exchange and (ii) hypercharge gauge boson exchange only [since at energies much higher than W/Z masses, we can revert back to using the W^a_μ, B_μ basis for gauge bosons and only the B_μ boson couples to right-handed electron] – this is just to make sure that the two approaches agree! Again, *final*-state particles are the charged (would-be) *NGB's* in both cases. The goal is to reproduce the result in Eq. 15.101 of Lahiri and Pal, which is obtained using *unitarity* gauge instead, i.e., (longitudinal) W's as final state particles.

Also, just like in problem 4.2, you can either use R_ξ gauge with finite ξ for obtaining couplings of Nambu-Goldstone bosons or (for a more "kosher" way) take the limit $v \rightarrow 0$ with gauge couplings held fixed (as I mentioned during lecture).

10 Homework 10 [Phenomenology of Standard Model (II)], due Monday, May 16 on ELMS

10.1 W boson decay

Calculate the branching ratio for W boson to decay into (all) leptons (i denotes generation index), i.e.,

$$\frac{\Gamma\left(W^+ \rightarrow l_i^+ \nu_i\right)}{\Gamma\left(W^+ \rightarrow \text{all}\right)} \tag{43}$$

You can generalize the result for Z boson decay width obtained in lecture (or section 15.4 of Lahiri and Pal) to that for W boson decays.

[9] Of course, one can imagine that there are *less* than three neutrinos, in which case part of the observed the invisible decay width of Z boson would have to come from some other final state, but such a particle is absent in the Standard Model.

You can neglect fermion masses relative to W boson mass (except of course for top quark which is actualy heavier than the W boson).

A part of the denominator above is the decay width of W boson into quarks for which be sure to include color factor and do *not* assume that CKM matrix is (even approximately) identity (but use the unitarity of the CKM matrix).

10.2 Top quark decay

(i) Calculate the decay width of the top quark (into bottom quark and W boson), assuming that $V_{tb} = 1$, $V_{ts,\,td} = 0$ and neglecting the bottom quark mass.

(ii) Calculate the decay width of top quark into bottom quark and the charged (would-be) Nambu-Goldstone boson, ϕ^+.

Just like in problems 4.2 and 9.5, you can use R_ξ gauge with finite ξ for obtaining couplings of Nambu-Goldstone boson. Equivalently, for a more "kosher" way, take the limit of gauge couplings $\to 0$ (i.e., *un*like in problems 4.2 and 9.5), but v and top quark Yukawa coupling held fixed.

(iii) Show that the above two results are the same in the (hypothetical) limit $m_t \gg m_W$.

Explanation: In the limit $m_t \gg m_W$ (i.e., energy of W boson is much larger than its mass), we can use the *equivalence theorem* to show that the coupling of top quark to *longitudinal* W boson is (approximately) same as the coupling to ϕ^+ and is thus much larger that the coupling of top quark to transverse W bosons. So, we expect the decay width of top quark into bottom quark and W boson to be dominated by the longitudinal polarization of W boson, i.e., into ϕ^+, as we find above. Of course, we can check this statement using the *explicit* longitudinal and transverse polarization vectors.

10.3 Higgs boson coupling to Z boson

(i) Show that there is *no* coupling of Z boson to two Higgs bosons starting from the Standard Model Lagrangian.

(As discussed in lecture, this means that a Higgs bosons can*not* be pair-produced via s-channel Z exchange in e^+e^- colliders.)

(ii) For the case of *on-shell* Higgs bosons (and irrespective of whether Z boson is on or off-shell), show that the above vanishing of the coupling follows simply from Bose-Einstein statistics and angular momentum conservation (i.e., with*out* really using the Standard Model Lagrangian).

10.4 Higgs boson decay into fermions

(i) Exercise 15.12 of Lahiri and Pal.

Start with the general formula for decay width of particle into two other particles (as in Eq. 7.9 of Lahiri and Pal). The result to be derived here is (basically) same as the one in Eq.

7.34 of Lahiri and Pal (and which might have been discussed in Phys 624 last semester), once we rewrite the Yukawa coupling of Higgs to leptons in terms of m_l, α etc. You are allowed to "consult" the derivation given in Lahiri and Pal, but of course do not simply copy it. You can also use part of the calculation of decay width of Z that is given in notes.

(ii) For a Higgs boson of mass 126 GeV (as discovered in 2012!), calculate the branching ratio for Higgs boson to decay into tau lepton.

Use the result given in Eq. 15.70 of Lahiri and Pal (that you derived above), generalizing it to the case of quarks, being careful about color factors as usual.

Use the fermion masses given below Eq. 12.111 of Lahiri and Pal. Can these masses be neglected in the last factor of Eq. 15.70?

(iii) Which is the dominant contribution to the *total* decay width of the Higgs boson (denominator of above branching ratio)?

(iv) Can the Higgs boson of this mass decay into W^+W^- or ZZ (does the required coupling exist)?[10]

10.5 Heavier (if you wish"hypothetical") Higgs boson decay into W^+W^-

(i) Calculate the decay width of such a Higgs boson (i.e., with mass larger than 2 M_W) into W^+W^- bosons.

(ii) Calculate the decay width of Higgs boson into the charged (would-be) Nambu-Goldstone bosons, i.e., $\phi^+\phi^-$.

Just like in problems 4.2, 9.5 and 10.2 (ii), you can use R_ξ gauge with finite ξ for obtaining couplings of Nambu-Goldstone boson. Equivalently, for a more "kosher" way, take the limit of gauge couplings $\rightarrow 0$ (i.e., *un*like in problems 4.2 and 9.5), but v and Higgs quartic coupling held fixed, i.e., like what was suggested as an alternative way in problem 10.2 (ii).

(iii) Show that the above two results are the same in the limit of energy of W, i.e., m_H being $\gg m_W$.

Explanation: similar to homework 10.2 above.

10.6 CP violation

Consider a process (which can be either a decay of a particle or a scattering of particles) whose total amplitude receives two contributions as follows:

$$\mathcal{A} \;=\; \mathcal{A}_1 + \mathcal{A}_2 \tag{44}$$

[10] Consider only *on*-shell W/Z's here and in next problem. There is also the channel (which is to be neglected here) with one W/Z being on-shell and the other one off-shell, with the off-shell one "decaying" into fermion-antifermion pair, i.e.,a net a *three*-body decay of Higgs.

where

$$\mathcal{A}_i = |\mathcal{A}_i| e^{i\delta_i} e^{i\phi_i} \tag{45}$$

where δ_i is a CP-violating (for example, from charged current weak interactions) phase (i.e., flips sign when we go to the CP-conjugate process) and ϕ_i is a CP-preserving (for example, from strong interactions) phase (which does not flip sign when we go to the CP-conjugate process). Denote the total amplitude for the CP-conjugate process by $\bar{\mathcal{A}}$. Calculate the CP-asymmetry defined by

$$A_{CP} \equiv \frac{|\mathcal{A}|^2 - |\bar{\mathcal{A}}|^2}{|\mathcal{A}|^2 + |\bar{\mathcal{A}}|^2} \tag{46}$$

In particular, you should find that in order to obtain a *non*-zero CP-asymmetry, *both* the CP-violating and the CP-preserving phases have to be different for the two components of the amplitude.