Choice of $\boxed{Z_1}$ (coefficient of counterterm (CT) for vertex function, Γ_μ)

- For on-shell fermions $\boxed{\text{and}}$ (very) small momentum transfer between them (q^ν), set the full vertex function (i.e., classical + loop + CT) to be same as classical value, i.e., loop is completely cancelled by CT (again, for this special choice of external momenta)

- Now, for on-shell fermions, we have

$$\Gamma_\mu^{\text{loop}} \underset{(\not p = \not p' = m)}{=} \gamma_\mu F_1^{\text{loop}}(q^2) + i\sigma_{\mu\nu} q^\nu F_2^{\text{loop}}(q^2) \quad \ldots (1)$$

$$\xrightarrow{q^\nu \to 0} \gamma_\mu F_1(0),$$

whereas CT gives $Q\gamma_\mu (Z_1 - 1)$

[Recall $\mathcal{L}_{CT}^{(c)} = -Qe(Z_1-1)\bar\psi \not A \psi$, giving $-ieQ(Z_1-1)\gamma_\mu$ for Feynman amplitude and $-ie\Gamma_\mu^{\text{loop}}$ is Feynman (loop) amplitude]

- Thus, setting sum of above 2 terms to zero gives

$$F_1(0) = -Q(Z_1 - 1)$$

- Next, consider general external momenta (i.e., fermions off-shell and $q^\nu \neq 0$). We would like to show (as follows) that divergence in this loop is also cancelled by $\boxed{\text{same}}$ choice of CT as above, although finite terms remain in this sum (i.e., loop is not completely cancelled by CT in this case)

— We have (in general, i.e., based on Lorentz and Dirac-space structure)

$$\Gamma_\mu(p, p') = \gamma_\mu f_1 + p_\mu \mathbb{1}_{4\times4} f_2 + p'_\mu \mathbb{1} f_3 + \sigma_{\mu\nu} p^\nu f_4 + \sigma_{\mu\nu} p'^\nu f_5 \quad \ldots (2)$$

(assuming parity invariance and dropping the superscript "loop")

- Here, by Lorentz invariance, f's are functions of p^2, p'^2 and $p \cdot p'$

- Also, divergence can only be in f_1, since "extraction" of (external) momentum in other terms reduces degree of divergence (from $D=0$ to $D=-1$, i.e, making it finite)

- For the special case of $p^2 = p'^2 = m^2$, Eq (2) reduces to Eq.(3), i.e., the "$\mathbb{1}_{4\times4}$" terms can be re-written in terms of γ_μ and $\sigma_{\mu\nu}$; similarly $p \cdot p'$ in terms of q^2 ⇒

[$q^2 = (p-p')^2 = 2m^2 - 2p \cdot p'$]

$$F_1(q^2) = f_1(\not p = \not p' = m) + \text{other } f\text{'s} \quad \ldots (3)$$

← not divergent

- Now, we can (Taylor) expand f_1 for general momenta around on-shell value, i.e.,

← still arbitrary

$$f_1(p^2, p'^2, p \cdot p') = f_1(p^2 = p'^2 = m^2, p \cdot p') + \left\{ \frac{\partial f_1}{\partial p^2}\bigg|_{p^2=m^2} \times (p^2 - m^2) + \frac{\partial f_1}{\partial p'^2}\bigg|_{p'^2=m^2} \times (p'^2 - m^2) + \ldots \right. \quad \ldots (4)$$

no divergence here, since $\partial/\partial p^2$ reduces it

⇒ As far as $\boxed{\text{divergence is concerned}}$, Eqs.(3) & (4) imply

$$f_1(\text{general momenta}) \text{"="} F_1(q^2) \ldots \overset{\text{expand}}{=} F_1(0) + q^2 \frac{dF_1}{dq^2}\bigg|_{q^2=0} + \ldots$$

divergence only here

— Thus, $\boxed{\text{divergence}}$ for general Γ_μ is same as in $F_1(0)$ and cancelled by (above) choice of CT