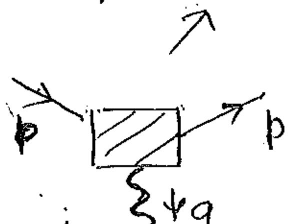
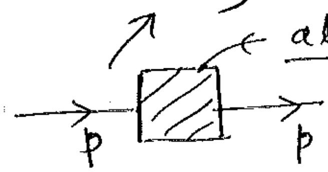
Proof of $\boxed{Z_1 = Z_2}$

- Start with the general (i.e., fermions & photon can be off-shell) Ward-Takahashi (WT) identity:

$$q_\mu \Gamma^{\mu,\text{loop}} = Q\left[\Sigma(p') - \Sigma(p)\right] \quad \cdots (1)$$



$p' = p - q$



all loops

Next

- Take the fermions to be on-shell ($\not p' = \not p = m$), where:

$$\Gamma_\mu^{\text{loop}}(p, p') = \gamma_\mu F_1^{\text{loop}}(q^2) + i\sigma_{\mu\nu} q^\nu F_2^{\text{loop}}(q^2) \quad \cdots (2)$$

(as shown in LP sec. 11.1 or PS sec. 6.2) assuming parity invariance (no γ_5)

- Furthermore, take limit of (very) small momentum transfer between fermions, i.e., $\boxed{q^\nu \to 0}$ (as when measuring charge in a static electric field: see LP sec. 11.2.1)
- So, LHS of Eq. (1) becomes $q_\mu \gamma^\mu F_1^{\text{loop}}(0)$ ... (3)
- On RHS of Eq. (1), do "Taylor expansion" [given $-q^\nu = (p'-p)^\nu \to 0$]:

$$Q \times \frac{\partial \Sigma}{\partial \not p}\bigg|_{\not p = m} \times (-\not q) \quad \cdots (4)$$

- So far, we are at level of loop calculation. Next, relate coefficients of counterterms to above loop result: (see LP Eq. 12.85)
  - Eq. (3) becomes $\not q(-Q)(Z_1 - 1)$ (see other note on choice of Z_1)
  - Eq. (4) becomes $+Q\not q \frac{\partial \Sigma_{CT}}{\partial \not p}\bigg|_{\not p = m}$, given the choice $\frac{\partial(\Sigma + \Sigma_{CT})}{\partial \not p}\bigg|_{\not p = m} = 0$

$\leftarrow$ see LP Eq. 12.84

$= +Q \not{q}\, a_{CT}$ (since $\Sigma_{CT} = a_{CT} \not{p} + b_{CT}$)

independent of p

$= -Q \not{q}\,(Z_2 - 1)$ (see just below LP Eq. 12.84)

- Equating these final LHS & RHS gives

  $Z_1 = Z_2$

- In HW, you are to calculate $Z_{1,2}$ <u>explicitly</u> (as opposed to general argument above), i.e., using
  - (i) DIMREG : you should indeed find $Z_1 = Z_2$ since this regularization preserves gauge invariance (hence WT identity)
  - (ii) hard cut-off : it does not respect gauge invariance so that $Z_1 \neq Z_2$ (actually, <u>divergent</u> pieces <u>are</u> equal, but <u>not</u> <u>finite</u> <u>terms</u>)