Decay width of $\boxed{Z}$ into $\boxed{\text{fermion - antifermion}}$

– Neglecting fermion mass, we have

$\overline{|\mathcal{M}|^2}$ [i.e., summed over spins of fermion (momentum p) and of antifermion (p') and averaged over polarizations of Z (momentum k)]

$$= \underbrace{\frac{\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{M_Z^2}\right)}{3}}_{\substack{\text{polarization}\\ \text{average}\\ \text{as in Eq. 15.64 of LP}}} \times \underbrace{\mathrm{Tr}\left[\not{p}\,\gamma^\mu (g_V - g_A\gamma_5) \times \not{p}' \times \gamma^\nu (g_V - g_A\gamma_5)\right]}_{\substack{\text{spin sum as in}\\ \text{Eq. 15.59 of LP with}}} \times \left(\frac{g_Z^2}{4}\right)$$

$$\mathcal{L}_{int} = -\frac{g_Z}{2}\bar{f}\gamma^\mu (g_V - g_A\gamma_5) f Z_\mu$$

– Next, above trace $= p_\rho p'_\sigma \times$

$$\mathrm{Tr}\Big[g_V^2(\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu) + g_A^2(\gamma^\rho\gamma^\mu\gamma_5\gamma^\sigma\gamma^\nu\gamma_5)$$
$$- g_V g_A(\gamma^\rho\gamma^\mu\gamma_5\gamma^\sigma\gamma^\nu + \gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu\gamma_5)\Big]$$

[Use $\mathrm{Tr}(\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu) = 4(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} - g^{\rho\nu}g^{\mu\sigma})$

and $\mathrm{Tr}(\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu\gamma_5) = 4i\varepsilon^{\rho\mu\sigma\nu}$, along with $\gamma_5^2 = \mathbb{1}$ and $[\gamma_5, \gamma_\mu] = 0$, i.e.,

$\gamma_5\gamma^\rho\gamma^\nu\gamma_5$ in 2nd term above $= (-1)^2\gamma^\rho\gamma^\nu\gamma_5^2$
$= +\gamma^\rho\gamma^\nu$

and similarly in 3rd, 4th terms above]

$$= 4\left(g_V^2 + g_A^2\right)\left(p^\mu p'^\nu + p^\nu p'^\mu - p\cdot p' g^{\mu\nu}\right)$$

[The $g_V g_A$ term gives zero since trace is antisymmetric in μ, ν vs. polarization sum being symmetric in μ, ν]

Thus, $$\overline{|\mathcal{M}|^2} = \frac{g_Z^2}{3}\left(g_V^2 + g_A^2\right) \times \left[p\cdot p' + \frac{2(p\cdot k)(p'\cdot k)}{M_Z^2}\right]$$

— Go to rest frame of Z, i.e., $k^\mu = (M_Z, \bar{0})$; $p^\mu = \left(\frac{1}{2}M_Z, \bar{p}\right)$ and $p'^\mu = \left(\frac{1}{2}M_Z, -\bar{p}\right)$, to give $$\overline{|\mathcal{M}|^2} = \frac{g_Z^2 M_Z^2}{3}\left(g_V^2 + g_A^2\right)$$

— To compute decay width, plug above $\overline{|\mathcal{M}|^2}$ into the general formula in Eq 7.9 of LP, i.e.,

$$\Gamma = \frac{1}{2M}\int |\mathcal{M}_{fi}|^2 \frac{d^3p}{(2\pi)^3(2E)} \frac{d^3p'}{(2\pi)^3(2E')} (2\pi)^4 \delta^4(k - p - p')$$

mass of decaying particle

in general, this depends on k, p, p'

—Now $\int d^3p' \, \delta^3(\bar{k} - \bar{p} - \bar{p}')$ simply sets $\bar{p}' = \bar{k} - \bar{p}$:

$$\Gamma = \frac{1}{2M}\int |\mathcal{M}_{fi}|^2 \frac{d^3p \; \delta(k_0 - p_0 - p_0')}{(2\pi)^2 \, 4EE'} \ldots$$ $\boxed{\text{evaluated}}$ for $\bar{p}' = \bar{k} - \bar{p}$,

energies

i.e., with $p_0 \equiv E = \sqrt{|\bar{p}|^2 + m_f^2}$ (similarly for $E' \equiv p_0'$) for decay into $f\bar{f}$ (i.e., particle - antiparticle pair)

- In rest frame of particle decaying, we have $k = (M, \bar{0})$ so that $\bar{p} = -\bar{p}'$ and $E = E'$:

$$\Gamma = \frac{1}{2M} \int |\mathcal{M}_{fi}|^2 \frac{d^3p}{4\pi^2\, 4E^2}\, \delta\left(M - 2\sqrt{|\bar{p}|^2 + m_f^2}\right)$$

Assuming $|\mathcal{M}_{fi}|^2$ is independent of direction of $\bar{p}$, the angular part of $\int d^3p$ gives 4π:

$$\Gamma = \frac{1}{2M} \int |\mathcal{M}_{fi}|^2 \frac{d|\bar{p}|\,|\bar{p}|^2}{4E^2\pi} \frac{\delta\left(|\bar{p}| - \sqrt{\frac{M^2}{4} - m_f^2}\right)}{\left|\frac{d}{d|\bar{p}|}\left(M - 2\sqrt{|\bar{p}|^2 + m_f^2}\right)\right|_{\text{at } |\bar{p}| = \sqrt{M^2/4 - m_f^2}}}$$

δ-function corresponding to energy conservation sets $|\bar{p}| = \sqrt{M^2/4 - m_f^2}$ and the denominator is the usual factor from change of variables in δ-function.

Thus, $$\Gamma = \frac{1}{2M} |\mathcal{M}_{fi}|^2 \times \frac{\left(M^2/4 - m_f^2\right)}{4 \times (M/2)^2 \pi} \frac{1}{2 \times \left.\frac{2|\bar{p}|}{2\sqrt{|\bar{p}|^2 + m_f^2}}\right|_{|\bar{p}| = \sqrt{\frac{M^2}{4} - m_f^2}}}$$

evaluated at $|\bar{p}| = \sqrt{M^2/4 - m_f^2}$

$$= \frac{|\mathcal{M}_{fi}|^2 \sqrt{M^2/4 - m_f^2}}{8 M^2 \pi}$$

Substituting $|\mathcal{M}_{fi}|^2$ from above gives (neglecting m_f)

$$\Gamma(Z \to f\bar{f}) = \frac{g_Z^2}{48\pi} M_Z \left(g_V^2 + g_A^2\right)$$