Notes on "converting" measure compensating function, $\Delta(A^\mu)$, into ghost fields

- Recall that

$$\text{(final) } Z[J] \sim \int \mathcal{D}A_\mu \, \delta[\mathcal{F}(A_\mu)] \, \Delta(A^\mu) \, e^{i\int d^4x \, \mathcal{L}_J(A_\mu)}$$

where $\boxed{\Delta(A^\mu)}$ (measure compensating function)

$$= \frac{1}{g} \times \left| \det\left[\left(\frac{\delta \mathcal{F}_a}{\delta A_b^\mu} \right) * D_{bd\,\mu}(A^\nu) \right] \right|$$

with $\boxed{D_{bd\,\mu}} = \delta_{bd}\partial_\mu - g f_{bcd} A_{c\mu}$ being the covariant derivative acting on a field in adjoint representation

- We will take (as an example) $\mathcal{F}_a(A^\mu) = \partial_\kappa A^\kappa_a$ (corresponding to Lorenz gauge condition familiar from canonical quantization of EM)

$$\Rightarrow \Delta(A^\mu) = \frac{1}{g} \times \left| \det\left(\underbrace{\delta_{ab}\,\partial_\mu}_{\text{from } \delta \mathcal{F}_a/\delta A_b^\mu} D^\mu_{bd} \right) \right|$$

$$= g^{-1} \left| \det\left(\delta_{ad} \Box - g f_{acd} A^\mu_c \partial_\mu \right) \right|$$

- As discussed in lecture, for abelian case (i.e., $f_{acd} = 0$), $\Delta(A^\mu)$ is <u>in</u>dependent of A_μ and thus factors out of $\int \mathcal{D}A_\mu$, i.e., is <u>ir</u>relevant...

- For non-abelian case, above $\Delta(A^\mu)$ does depend on A_μ ... $\Rightarrow$ cannot be factored out...

- Since $\Delta(A_\mu)$ is a determinant, it can be written as functional integral for complex scalar field (in adjoint representation of gauge group), but with anti-commutation relations:

$$\Delta(A^\mu) \sim \prod_{a=1\ldots n^2-1} \int \mathcal{D}\bar{\eta}_a \int \mathcal{D}\eta_a \, e^{-i\int d^4x\, \bar{\eta}_b M_{bc} \eta_c}$$

where $$M_{bc} = \frac{\delta \mathcal{F}_b}{\delta A^\mu_d} D^\mu_{dc}(A^\nu)$$

[Use (determinant) form of $\Delta(A^\mu)$ and the fermion (i.e., anti-commuting field) functional integral formula:

$$\int \mathcal{D}\bar{\Psi} \int \mathcal{D}\Psi \, e^{i\int d^4x\, \bar{\Psi} A \Psi} \sim (\det A)^{+1}$$

(vs. $1/\det A$ for commuting fields)

- Thus, $$Z[J] \sim \prod_{\substack{a=1\ldots n^2-1 \\ \mu=0\ldots 3}} \int \mathcal{D}A^\mu_a \int \mathcal{D}\bar{\eta}_a \int \mathcal{D}\eta_a \, e^{i\int d^4x\, \mathcal{L}^{(\text{eff})}_J}$$

where $$\mathcal{L}^{(\text{eff.})}_J = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a - \underbrace{(\bar{\eta} M \eta)}_{\text{from } \Delta(A^\mu) \text{ as above}} - \sum_b [\mathcal{F}_b(A^\nu)]^2/(2\xi) - \sum_a J^\mu_a A^a_\mu$$

– For the choice $\mathcal{F}_a(A^\nu) = \partial_\kappa A^\kappa_a$, we have

$$M_{ac}\,\eta_c = \left[\frac{\delta \mathcal{F}_a}{\delta A_b{}^\mu}\, D^\mu_{bc}(A^\nu)\right]\eta_c$$

$$= \delta_{ab}\,\partial_\mu\left(\delta_{bc}\,\partial^\mu - g f_{bdc} A^\mu_d\right)\eta_c$$

$$= \partial_\mu\left(\partial^\mu \eta_a - g f_{adc} A^\mu_d \eta_c\right)$$

acts here

so that the addition to the action is

(n^2-1) vector

$$-\int d^4x\, \bar{\eta}\, M\, \eta = -\int d^4x\, \bar{\eta}_a M_{ac}\, \eta_c$$

$$= \int d^4x\, \left(\partial_\mu \bar{\eta}_a\right)\left(\partial^\mu \eta_a - g f_{adc} A^\mu_d \eta_c\right)$$

acts only here (integration by parts)

$$= \int d^4x\; 2\,\mathrm{tr}\left\{\partial_\mu \bar{\eta}\left(\partial^\mu \eta + ig\left[A^\mu, \eta\right]\right)\right\}$$

where in last line, both η and A^μ are written in $(n^2-1)\times(n^2-1)$ matrix form, i.e.,

$\eta = \sum\limits_{a=1\ldots n^2-1} T^a \eta_a$ (T^a being generator in fundamental representation)

(usual) gauge fixing term

– Thus, $$\mathcal{L}^{eff}_J = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a - \frac{\mathrm{tr}}{\xi}\left(\partial_\mu A^\mu\right)^2 - \sum_a J^\mu_a A^a_\mu$$

$$+ 2\,\mathrm{tr}\left\{\left(\partial_\mu \bar{\eta}\right)\left(\partial^\mu \eta + ig\left[A^\mu, \eta\right]\right)\right\}$$

gives η propagator

gives interaction of η with A^μ (again, ∂_μ here acts only on $\bar{\eta}$)

- Once we have "read-off" $\mathcal{L}^{\text{eff}}_J$ from the exponent in functional integral expression, we can "forget" how it was derived and just use $\mathcal{L}^{\text{eff}}$ (with<u>out</u> source term) for calculating Feynman diagrams (as in <u>canonical</u> quantization)...

[To be more accurate, once integrand is of exponential form, (functional) integration can be carried out just like for scalar/fermion fields ... obtaining correlation functions by $\delta/\delta J$ ...]

- Just to show clearly that ghosts are basically a "trick", one can show (see HW5) that there is a gauge where ghost fields decouple (even for <u>non</u>-abelian case)

- And, this trick can be applied to other cases in order to convert determinant factor in functional integral into exponential (and hence extra terms/fields in effective Lagrangian)