Notes on general formula for computing transition amplitudes via path integral (PI)

(mostly from Peskin & Schroeder, pages 280 - 281)

- Consider arbitrary set of coordinates q^i, their conjugate (canonical) momenta p^i and arbitrary Hamiltonian, $H(q, p)$

- Compute $\langle q_{final} | e^{-iH(t_{final} - t_{initial})} | q_{initial} \rangle$ by breaking (finite) time interval into N infinitesimal intervals (each of duration Δt), i.e., $e^{-iH(t_{final} - t_{initial})} = e^{-iH\Delta t} e^{-iH\Delta t} \cdots e^{-iH\Delta t}$ and

insert complete set of intermediate coordinate states (between each of the $e^{-iH\Delta t}$ factors), i.e., $\mathbb{1} = \prod \int dq_k^i \, |q_k\rangle \langle q_k|$, where ← drop "i" here for simplicity

i labels different q's (e.g., particles)

$k = 1 \ldots (N-1)$ denotes which time slice

- Thus, we get product of N factors of form

$$\langle q_{k+1} | e^{-iH\Delta t} | q_k \rangle \approx \langle q_{k+1} | (1 - iH\Delta t) | q_k \rangle$$

← again, several q's (e.g., particles) here in general ... but suppress that index

[with $q_0 = q_{initial}$, $q_N = q_{final}$]

- Consider a term $f(q)$ ← no p in H: obviously

$$\langle q_{k+1} | f(q) | q_k \rangle = f(q_k) \prod_i \delta(q^i_k - q^i_{k+1})$$

← "restore" ...

rewrite (see later for reason)

$$= f\left(\frac{q_{k+1} + q_k}{2}\right) \prod_i \underbrace{\int \frac{dp^i_k}{2\pi} \exp\left[i \sum_i p^i_k (q^i_{k+1} - q^i_k)\right]}_{\text{"representation" of } \delta\text{-function}}$$

- Next, $g(p)$ ← no q in H: introduce complete set of momentum eigenstates so that

$$\langle q_{k+1} | g(p) | q_k \rangle = \prod_i \int \frac{dp^i_k}{2\pi} g(p_k) \exp\left[i \sum_i p^i_k (q^i_{k+1} - q^i_k)\right]$$

use $\langle q_{k+1} | p_k \rangle = e^{+i p_k \cdot q_{k+1}}$ (like $\langle x | p \rangle = e^{i p \cdot x}$ for 1 particle)

[that's why $\langle q_{k+1} | f(q) | q_k \rangle$ was re-written above...]

$\Rightarrow$ If $H = g(p) + f(q)$, then

$$\langle q_{k+1} | \underbrace{H (q, p)}_{\text{operators}} | q_k \rangle = \prod_i \int \frac{dp^i_k}{2\pi} H\left(\frac{q_{k+1} + q_k}{2}, p_k\right) \times \exp\left[i \sum_i p^i_k (q^i_{k+1} - q^i_k)\right]$$

c-numbers

- On the other hand, if H contains products of p, q, then (in general) above formula is not true since order of p, q matters on left-hand-side (LHS) — where H is operator — but not on RHS [H is (function of) c-number(s)]

- However, above formula is true for Weyl-ordered H, i.e., q's appearing symmetrically on left and right, e.g.,

$$\langle q_{k+1} | \frac{1}{4}\left(q^2p^2 + 2qp^2q + p^2q^2\right) | q_k \rangle = \left(\frac{q_{k+1}+q_k}{2}\right)^2 \times \langle q_{k+1} | p^2 | q_k \rangle .$$

(explicitly, act the q's appearing to left in H on q_{k+1} ... to get above formula ... then, use above manipulation for "matrix element" of "remaining" $(p^2 \ldots)$)

- Any Hamiltonian can be put into this form by commuting p's and q's ... generating extra terms in this process ... which must appear on RHS of above formula

$\Rightarrow$ assuming H is Weyl-ordered,

$$\langle q_{k+1} | e^{-i\Delta t H} | q_k \rangle = \prod_i \int \frac{dp_k^i}{2\pi} \exp\left[i \sum_i p_k^i (q_{k+1}^i - q_k^i) \right] \times \exp\left[-i\Delta t H\left(\frac{q_{k+1}+q_k}{2}, p_k\right) \right]$$

"back to" exponential form (again, ε is small)

- Above was infinitesimal time factor: multiply N such factors and integrate over intermediate coordinates to obtain

$$\langle q_{final}(=q_N) | e^{-iH(t_{final}-t_{initial})} | q_{initial}(=q_0)\rangle$$

$$= \prod_{i,k} \int dq^i_k \int \frac{dp^i_k}{2\pi} \exp\left\{ i \sum_k \left[\sum_i p^i_k (q^i_{k+1} - q^i_k) - \Delta t H\left(\frac{q_{k+1}+q_k}{2}, p_k \right) \right] \right\}$$

i: e.g., different particles; k: different time slices

- Since there is 1 $\int dp$ for each k (again, time-slice) from 0 to $(N-1)$ and 1 $\int dq$ for each k from 1 to $N-1$, the continuum form of above formula is

$$\prod_i \mathcal{D}q(t)\,\mathcal{D}p(t) \exp\left\{ i \int_{t_{initial}}^{t_{final}} dt \left[p^i \dot{q}^i - H(q,p) \right] \right\}$$

with $q(t)$ constrained at endpoints ($t = t_{initial}, t_{final}$) to be $q_{initial,\, final}$, but $p(t)$'s are arbitrary

[Again, $\int \mathcal{D}q(t)\,\mathcal{D}p(t)$ is really $\prod_i \int \frac{dp^i dq^i}{2\pi}$ at each point in time]

- Even though the integrand in exponent "looks like" Lagrangian (and thus exponent like action), in general [e.g., with $L = \dot{q}^2/2\, f(q)$ — see Ryder, pages 163–164] we don't quite get it in the end (i.e., after $\int dp$)
- Of course, for $H = p^2/(2m) + V(x)$, we can evaluate (Gaussian) p-integral to obtain action in exponent...