Notes on the "measure compensating" function, $\Delta(A^\mu)$, which appears in generating functional for gauge fields as

$$Z_J \sim \int \mathcal{D} A_\mu \, \delta[\mathcal{F}(A_\mu)] \, \Delta(A^\mu) \, \exp\left[i\int d^4x \, \mathcal{L}_{J}(A_\mu)\right]$$

source

- Begin with its general form
- "Partition" the functional integral into a component $\int \mathcal{D}\bar{A}_\mu$, which includes only those configurations which are not related by gauge transformations, and a component $\int \mathcal{D}\theta$, which denotes all possible gauge transformations (dropping the $a = 1 \ldots n^2-1$ index on both A_μ and θ for simplicity), i.e.,

(trial) $$Z_J \sim \int \mathcal{D}\bar{A}_\mu \int \mathcal{D}\theta \, \delta\left[\mathcal{F}(\bar{A}^{\theta}_\mu)\right] e^{\left[i\int d^4x \, \mathcal{L}_J(\bar{A}^{\theta}_\mu)\right]}$$

drop $\prod_{\mu=0\ldots3}$ as well for simplicity

where $\mathcal{L}_J(A_\mu) = -J^a_\mu A^\mu_a - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a$

and $\delta[\mathcal{F}(A_\mu)]$ in functional integral enforces the (gauge-fixing) condition $\mathcal{F}(A_\mu) = 0$ [e.g., $\mathcal{F}(A_\mu) = \partial_\nu A^\nu$ gives Lorenz gauge]

Finally, $\bar{A}_\mu^\theta$ denotes gauge-transformed version of $\bar{A}_\mu$

[Basically, for given $\bar{A}_\mu$, we end up "picking" from $\int \mathcal{D}\theta$ only the θ which satisfies $\mathcal{F}(\bar{A}_\mu^\theta) = 0$]

- Since action is gauge-invariant, we can replace $\bar{A}_\mu^{\theta}$ by $\bar{A}_\mu$ there to get

$$\sim \underbrace{\int \mathcal{D}\bar{A}_\mu}_{\text{again, gauge-inequivalent configurations only (as desired)}} \underbrace{\exp\left[i\int d^4x\, \mathcal{L}_J(\bar{A}_\mu)\right]}_{\text{no } \theta \text{ here}} \times \underbrace{\int \mathcal{D}\theta\; \delta\left[\mathcal{F}(\bar{A}_\mu^\theta)\right]}_{\text{define this to be } 1/\Delta(\bar{A}^\mu)}$$

(functional)

So, we need to "re"-weight integrand by above $\Delta(\bar{A}^\mu)$ so that each representative (of gauge-equivalent class) A_μ is picked with equal weight

- Note that $\Delta(\bar{A}^\mu)$ is gauge-invariant, i.e., $\Delta(\bar{A}_\mu^{\theta^{(1)}}) = \Delta(\bar{A}^\mu)$, since combination of two gauge transformations [$\theta^{(1)}$ here and θ in definition of $\Delta(\bar{A}_\mu)$] is another gauge transformation... and we are integrating over all gauge transformations in Z_J

$$\Rightarrow \text{(final) } Z_J \sim \int \mathcal{D}\bar{A}_\mu \, \mathcal{D}\theta \, \delta\left[\mathcal{F}(\bar{A}_\mu^\theta)\right] \times \Delta\left[\bar{A}_\mu^\theta\right]$$

$$\times \exp\left[i \int d^4x \, \mathcal{L}_J(\bar{A}_\mu^\theta)\right]$$

new (re-weighting) factor (relative to trial Z_J),

but with $\bar{A}_\mu$ "replaced" by $\bar{A}_\mu^\theta$ due to $\Delta(\bar{A}_\mu)$ being gauge-invariant

- Relabeling $\bar{A}_\mu^\theta$ by A_μ [i.e., "going back" to integrating over all configurations] gives

$$Z_J \sim \int \mathcal{D}A_\mu \, \delta\left[\mathcal{F}(A_\mu)\right] \Delta(A^\mu) \exp\left[i \int d^4x \, \mathcal{L}_J(A_\mu)\right]$$

- Next, calculate $\Delta(A_\mu) = 1 \Big/ \int \mathcal{D}\theta \, \delta\left[\mathcal{F}(A_\mu^\theta)\right]$

- Infinitesimal gauge transformations are

$$A_\mu^\theta{}_b(x) = A_{\mu b}(x) + f_{bcd} A_{\mu c}(x) \, \theta_d(x) - \frac{1}{g} \delta_{bd} \partial_\mu \theta_d(x)$$

[and $\psi^\theta(x) = \psi(x) + i \, \theta_a(x) T^a \psi(x)$] ← again, gauge transformed ψ

→ generator of representation of ψ

Note that the above transformation are related to the ones used in lecture / Lahiri & Pal Eq. 14.11 :

$$\psi' = \psi - i g \beta_a(x) T_a \psi \quad \text{and}$$

$$A'^a_\mu = A^a_\mu + \partial_\mu \beta_a + g f_{abc} \beta_b(x) A^c_\mu$$

provided we identify β_a with $-\frac{1}{g}\theta_a$

– In these notes, we'll use the θ notation above (instead of β of lecture) – I am sorry for this change, but (hopefully) it's a "trivial" one!

Thus, $\mathcal{F}_a\left(A^{\theta}_{\mu b}\right) \approx \mathcal{F}_a(A_{\mu b}) + \frac{\delta \mathcal{F}_a}{\delta A_{\mu b}} \times \underbrace{\left[f_{bcd} A_{\mu c} - \frac{1}{g}\delta_{bd}\partial^{\mu}\right]\theta_d(x) \times \delta^{(4)}(x-y)}_{\text{shift in } A_{\mu b}}$

← functional derivative

since we have n^2-1 conditions, e.g., $\partial_\nu A^\nu_a = 0$ for each $a = 1 \ldots n^2-1$

– For a compact notation, define

$$\boxed{D_{bd\mu}} \equiv \delta_{bd}\partial_\mu - g f_{bcd} A_{\mu c} \quad \text{so that}$$

$$\underline{\text{shift}} \text{ in } \mathcal{F}_a(A^\nu) = \underbrace{\left[-\frac{1}{g}\left(\frac{\delta \mathcal{F}_a}{\delta A_{b\mu}}\right) \times D_{\mu bd}(A^\nu)\right]}\theta_d(x)\, \delta^{(4)}(x-y)$$

↖ this is just "response" of $\mathcal{F}_a(A^\nu)$ to an infinitesimal gauge transformation [with parameter $\theta(x)$]

Thus, in analogy to

$$\int dx\, \delta[f(x)] = \frac{1}{f'(a)} + \ldots \quad \text{where } f(a) = 0$$

← like above response...

we get

$$\prod_a \int \mathcal{D}\theta_a\, \delta\left[\mathcal{F}_a(A_{b\mu})\right] = g \Big/ \left|\det\left[\left(\delta\mathcal{F}_a/\delta A_{\mu b}\right) D_{\mu bd}(A^\nu)\right]\right|$$

↑ δ-function

where "det" is in adjoint representation, i.e., $a = 1 \cdots n^2-1$, i.e., in "operator" space and in the sense of functional integral

[e.g. $\int \mathcal{D}\phi \exp\{i\int d^4x\,(\phi A \phi)\} \sim 1/\sqrt{\det A}$]

$$\Rightarrow \Delta(A^\mu) = g^{-1}\left|\det\left[\left(\delta \mathcal{F}_a / \delta A_{\mu b}\right) D_{\mu bd}(A^\nu)\right]\right|$$

- We can "prove" above formula (why "det"?) by diagonalizing (in both spaces mentioned above) ... like we did earlier for case of scalar field functional integral ...

- start with abelian case, i.e., only one $\theta(x)$: we want to show that

$$\int \mathcal{D}\theta\, \delta(B\theta) \sim 1/\det B,$$

where δ is δ-function, where B is some hermitian operator [$(\delta \mathcal{F}/\delta A) D$ should be chosen as hermitian in order to have hermitian Lagrangian density for ghost fields: see later]

- As we did for scalar field, expand the arbitrary $\theta(x)$ in complete/orthonormal set of eigenfunctions of B: $\theta(x) = \sum_j c_j \theta_j(x)$, where $B\,\theta_j(x) = b_j\,\theta_j(x)$, b_j eigenvalue

so that $\int \mathcal{D}\theta(x) \sim \prod_j \int dc_j \quad \cdots (1)$

(just like for $\int \mathcal{D}\phi$ case done earlier)

- Thus, we can have a precise definition of δ-function of functional as follows:

$\delta[B\theta(x)]$ implies only non-vanishing contribution is from $B\theta(x) = 0$ for all x, i.e.,

$$B\left[\sum_j c_j \theta_j(x)\right] = \sum_j c_j b_j \theta_j(x) = 0$$

Thus all of the coefficients of $\theta_j(x)$ must be zero, i.e., $\delta B[\theta(x)] = \prod_j \delta(c_j b_j) \quad \cdots (2)$

↳ ordinary δ-function

Combining (1) & (2), we get

$$\int \mathcal{D}\theta\, \delta(B\theta) \sim \prod_j \int dc_j \underbrace{\delta(c_j b_j)}_{\delta(c_j)/b_j} = \frac{1}{b_1 \cdots b_\infty} = \frac{1}{\det B}$$

— Next, generalize to non-abelian case, i.e., consider $\prod_{a=1\ldots n^2-1} \int \mathcal{D}\theta_a \prod_b \delta(B_{bc}\theta_c)$ since adjoint representation, i.e., both θ and B come with $a(=1\ldots n^2-1)$ indices

- We need to satisfy $B_{bc}\theta_c(x) = 0$ for $\boxed{\text{all}}$ b
- diagonalize operators B_{bc} in adjoint representation space, i.e.,

$B_{bc} \to B^{(1)}_b \delta_{bc}$ ← new basis so that $\underbrace{B^{(1)}_b}_{\text{not matrix}} \theta^{(1)}_b(x) = 0$

i.e., for each b, we can repeat the above abelian story (and functional integral over θ_a in original basis is same as over $\prod_a \theta^{(1)}_a$ ...

still matrix in operator space

Finally we obtain, $\int \mathcal{D}\theta_a \prod_b \delta(B_{bc}\theta_c) = (\det B^{(1)}_1)^{-1} \cdots (\det B^{(1)}_{n^2-1})^{-1}$

$= 1/(\det B)$, where "det" is in both spaces ...