Notes on (functional integral formulation of fermion fields

- (Quantum) Fermion/Dirac fields satisfy anti-commutation relations ($\Rightarrow$ Fermi-Dirac statistics / Pauli exclusion principle obeyed by the corresponding quanta/particles)
- In order to incorporate this feature in PI formalism, we need "classical" fermion fields, i.e., "anti-commuting" c-number functions
- We can start with anti-commuting numbers (also called Grassmann numbers/algebra) and use them to "build" anti-commuting functions (see below)

Grassmann algebra

- Let ξ be an anticommuting number such that

(i) $\xi^2 = 0$ (ii) $\int d\xi = 0$ (iii) $\int \xi \, d\xi = 1$ and

(iv) $\dfrac{df(\xi)}{d\xi} = f'(0)$, where $f'(x)$ is 1^{st} derivative of $f(x)$

$$= \int f(\xi)\, d\xi$$

[Note that Grassmann numbers cannot be put in order (from "small" to "large"). Thus $\int d\xi$ means sum over all Grassmann numbers.]

Explanation of above formulae

- (i) follows simply from anticommutativity: $[\xi, \xi]_+ \equiv \xi\xi + \xi\xi = 2\,\xi\xi = 0$

- (ii) and (iii) are only non-trivial and consistent definition of integrals as follows...

Shift the integral variable in $\int \xi\, d\xi$ by a Grassmann number. Since integral is really sum over all Grassmann numbers, it should not change, i.e.,

$$\int (\xi + \eta)\, d(\xi + \eta) = \int (\xi + \eta)\, d\xi = \int \xi\, d\xi \Rightarrow \eta \int d\xi = 0$$

... for any $\eta \Rightarrow \int d\xi = 0$, i.e., (ii) above

Suppose we set $\int \xi\, d\xi = 0$ [instead of 1 as in (iii) above]. Then $\int d\xi = 0 = \int \xi\, d\xi$. Since $\int \xi^{N (\geq 2)}\, d\xi = 0$ by (i), we would get that all integrals are zero (trivial) $\Rightarrow$ only non-trivial definition of integral is to choose $\int \xi\, d\xi \neq 0$ ($= 1$ by choice of normalization).

- Use (i) - (iii) to prove (iv) as follows. Taylor expand: $f(\xi) = f(0) + f'(0)\,\xi + \frac{1}{2!}\xi^2 f''(0) \ldots$ where 3rd term onwards is zero due to (i)

Thus $df(\xi)/d\xi = f'(0)$. Finally, use (ii) & (iii) and above Taylor expansion of $f(\xi)$ in $\int f(\xi)\, d\xi$: only $f'(0)\,\xi$ term in expansion of $f(\xi)$ contributes to integral, giving $f'(0)$.

In order to describe complex (non-hermitian) fermion fields (i.e., $\psi^{\dagger} \neq \psi$) (see below), we need a set of two Grassmann numbers like ξ above. Denote them by $\zeta \equiv \xi + i\eta$ and $\bar{\zeta} = \xi - i\eta$.

Then above relations for ξ (and similarly η) give

(i) $\zeta^2 = 0 = \bar{\zeta}^2$ and $\zeta\bar{\zeta} + \bar{\zeta}\zeta = 0$

(ii) $\int d\zeta = 0$ and $\int d\bar{\zeta} = 0$

(iii) $\int \bar{\zeta}\, d\zeta = \int \zeta\, d\bar{\zeta} = 2$ and $\int \zeta\, d\zeta = 0 = \int \bar{\zeta}\, d\bar{\zeta}$

and $\int \bar{\zeta}\zeta\, d\bar{\zeta}\, d\zeta = +4$

Fermion/Dirac fields

- A Grassmann field is a function of space time whose values are anti-commuting numbers:

$$\tilde{\phi}(x) = \sum_i \underbrace{\xi_i}_{\text{Grassmann numbers}} \underbrace{\phi_i(x)}_{\text{ordinary (c-number) basis functions}}$$

for Grassmann

- In particular, to describe fermion/Dirac field, we take $\phi_i(x)$ above to be a basis of 4-component spinors, i.e., complete set of eigenfunctions (Dirac spinors) of a hermitian operator, A:

$A\,\psi_i(x) = a_i\,\psi_i(x)$ and $\bar{\psi}_i(x)\,A = \bar{\psi}_i(x)\,a_i$ $(i = 1, 2 \ldots \infty)$

with $\underbrace{\psi(x)}_{\text{fermion/Dirac field}} = \sum_j \zeta_j\,\psi_j(x)$ and thus $\bar{\psi}(x) = \sum_j \bar{\psi}_j(x)\,\bar{\zeta}_j$

In PI formalism, we have to evaluate functional integral of

ξ_j is not spinor $\Rightarrow$ "bar" is just complex conjugate

$\exp\left(i\int d^4x\, \bar{\Psi} A \Psi\right)$ ... which using above relations is $\exp\left(\sum_j i a_j \bar{\xi}_j \xi_j\right) = \prod_j \left(1 + i a_j \bar{\xi}_j \xi_j\right)$

[explicitly, Taylor-expand the exponential and use $\xi^2 = 0$ etc...]

Thus, $\int \mathcal{D}\bar{\Psi} \int \mathcal{D}\Psi \exp\left(i\int d^4x\, \bar{\Psi} A \Psi\right)$

$$= \prod_j \int d\bar{\xi}_j\, d\xi_j \left(1 + i a_j \bar{\xi}_j \xi_j\right) = \prod_j \left(+ 4 i a_j\right)$$

use above properties of integrals

$\sim (\det A)$

... cf $1/(\det A)$ for a complex scalar field ...

(We will use above result to "derive" ghost fields abit later.)

Just like for scalar field discussed in lecture, we can (re-) derive the Feynman rules for Dirac theory by using PI expression for generating functional [with source field being Grassmann-valued]: see Peskin & Schroeder, page 302 - 303