# Notes for (Review of) Heisenberg vs. Interaction picture

- In short, perturbation theory is usually developed in interaction picture (I.P.), where operators obey free equations of motion (i.e., they carry time-dependence due to free Hamiltonian) and state vectors carry time-dependence stemming from interactions part of Hamiltonian

vs.

Heisenberg picture (H.P.), where operators carry all of time-dependence, i.e., arising from full Hamiltonian

- Start with $\boxed{\text{I.P.}}$: transition amplitude given by $S_{fi} \equiv \langle f | \mathbf{S} | i \rangle$, where S-matrix is given by

$$S \equiv U(\infty, -\infty) \quad , \text{with}$$

$$U(t, t_0) = \sum_n (-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{n-1}} dt_n \, H_{int}(t_1) \ldots H_{int}(t_n)$$

interactions part of Hamiltonian, written in terms of fields in I.P.

$$= \sum_n \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n \times$$

vs. $t_{n-1}$ in above expression

time-ordered $\leftarrow$ $T\left[ H_{int}(t_1) \cdots H_{int}(t_n) \right]$

(Of course, $H_{int} = \int d^3x \, \mathcal{H}_{int}$ $\leftarrow$ density)

Operators obey $\dfrac{dO^{(I)}}{dt} = i\left[\underset{\uparrow \text{ free part}}{H_0^{(I)}}, O^{(I)}\right]$ in I.P.

[assuming no explicit time-dependence in $O^{(I)}$]

State vectors obey $\dfrac{d|\psi^{(I)}\rangle}{dt} = -i H_{int}^{(I)} |\psi^{(I)}\rangle$

i.e., the above time-evolution operator $U(t, t_0)$ obeys $\dfrac{dU(t,t_0)}{dt} = -i H_{int}^{(I)}(t)\, U(t,t_0)$ with "boundary condition" $U(t_0, t_0) = 1$

and $U^{+}(t,t_0)\, U(t,t_0) = \mathbb{1}$ (unitary)

[above 2 equations imply $U^{+}(t,t_0) = U(t,t_0)^{-1} = [U(t_0,t)]$

and $U(t,t_1)\, U(t_1,t_0) = U(t,t_0)$]

- Go to $\boxed{\text{H. P.}}$ from I.P. by unitary transformation:

Heisenberg picture

$$O^{(H)}(t) = U^{+}(t,0)\; O^{(I)}(t)\; U(t,0)$$

[assume $t_0 = 0$ without loss of generality]

so that $\dfrac{dO^{(H)}}{dt} = i\,[\underbrace{H_0^{(H)} + H_{int}^{(H)}}_{\text{full Hamiltonian}}, O^{(H)}]$

and $|\psi^{(H)}(t)\rangle = U^{+}(t,0)\, |\psi^{I}(t)\rangle$ so that $\dfrac{d|\psi^{(I)}(t)\rangle}{dt} = 0$

- Define $|in\rangle$ and $|out\rangle$ states in H.P. (according to Lehmann - Symanzik - Zimmermann):

$$|\alpha^{in}\rangle \equiv |\alpha(t=-\infty)^{(H)}\rangle = U^{+}(-\infty,0)\,|\alpha(t=-\infty)^{(I)}\rangle$$

like the states $b^{+}_{\dots}$ or $a^{+}_{\dots}|0\rangle$ that are used in (usual) perturbation theory calculations

$$|\beta^{out}\rangle \equiv |\beta(t=+\infty)^{(H)}\rangle = U^{+}(+\infty,0)\,|\beta(t=+\infty)^{(I)}\rangle$$

so that $S_{\beta\alpha} = \langle \beta | S | \alpha \rangle$

$$= \langle \beta(t=+\infty)^{(I)} |\, U(+\infty,-\infty)\, | \alpha(t=-\infty)^{(I)}\rangle$$

$$= \langle \beta(t=+\infty)^{(I)} |\, U(\infty,0)\ \underbrace{U(0,-\infty)}_{U^{+}(-\infty,0)}\, | \alpha(t=-\infty)^{(I)}\rangle$$

$$= \langle \beta^{out} | \alpha^{in}\rangle$$

$\boxed{\text{Matrix element}}$ of local operator

– In I.P. (with all higher-order corrections included), we have

$$\langle O(\bar{x},t)\rangle_{\beta\alpha} \equiv \langle \beta(t=+\infty)^{(I)} |\, U(\infty,t)\, O^{(I)}(\bar{x},t)\, U(t,-\infty)\, | \alpha(t=-\infty)^{(I)}\rangle$$

[ Note that $O^{I}(\bar{x},t)$ <u>already</u> includes time-dependence coming from <u>free</u> part of Hamiltonian... so, we have to "add" time-dependence from interactions in the form of $U$'s ]

$$= \langle \beta(t=+\infty)^{(I)} |\, U(\infty,0)\, \underbrace{U(0,t)}_{U^{+}(t,0)}\ O^{(I)}(\bar{x},t)\, U(t,0)\, \underbrace{U(0,-\infty)}_{U^{+}(-\infty,0)} |$$

$$= \langle \beta^{out} |\, O^{(H)}(\bar{x},t)\, | \alpha^{in}\rangle$$

Two-point correlation / Green's function / Feynman propagator

$i\Delta_F(x-y)\big|_{\text{all orders}} =$

Feynman

$\langle 0(t=\infty)^{(I)}| U(+\infty, t_x)\,\phi^{+(I)}(\bar{x}, t_x)\, U(t_x, t_y)\,\phi^{(I)}(\bar{y}, t_y) \times$ I.P.

vacuum state

$U(t_y, -\infty)\,|0(t=-\infty)^{(I)}\rangle \times \theta(t_x - t_y)$

time-ordered

$+ \langle 0(t=\infty)^{(I)}| U(+\infty, t_y)\,\phi^{(I)}(\bar{y}, t_y)\, U(t_y, t_x)\,\phi^{+(I)}(\bar{x}, t_x) \times$

$U(t_x, -\infty)\,|0(t=-\infty)^{(I)}\rangle \times \theta(t_y - t_x)$

[Again, $\phi^{(I)}(\bar{x}, t)$ includes time-dependence due to free Hamiltonian so that time-dependence due to interactions has to be added via U's...]

Using $U(\infty, t_x) = U(\infty, 0)\, U(0, t_x)$,

$U(t_x, t_y) = U(t_x, 0)\, U(0, t_y)$ and $U(t_y, -\infty) = U(t_y, 0) \times U(0, -\infty)$

and relation between I.P & H.P. fields and I.P. and $|{}^{\text{in}}_{\text{out}}\rangle$ states gives (for 1st time-ordering above) $\langle 0^{\text{out}}|\,\phi^{+(H)}(\bar{x}, t_x)\,\phi^{H}(\bar{y}, t_y)\rangle|0^{\text{in}}\rangle\ldots$

Similarly for other time ordering... so that

$i\Delta_F(x-y)\big|_{\text{all orders}} = \langle 0^{\text{out}}|\,T\left[\phi^{+(H)}(\bar{x}, t_x)\,\phi^{(H)}(\bar{y}, t_y)\right]|0^{\text{in}}\rangle$ H.P.

... which can be extended to N-point Green's function

$\langle 0(t=\infty)^{(I)}|\,T\left[U(\infty, -\infty)\,\phi^{(I)}(x_1)\cdots\phi^{(I)}(x_N)\right]|0(t=-\infty)^{(I)}\rangle$ I.P.

$= \langle 0^{\text{out}}|\,T\left[\phi^{(H)}(x_1)\cdots\phi^{(H)}(x_N)\right]|0^{\text{in}}\rangle$