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Deep Inelastic Electron - Proton Scattering :

from general formula (for cross-section) to more convenient (and specific to QED) one (based on Peskin & Schroeder: section 17-3)

— Recall that the general formula for cross-section for electron-proton inelastic scattering is

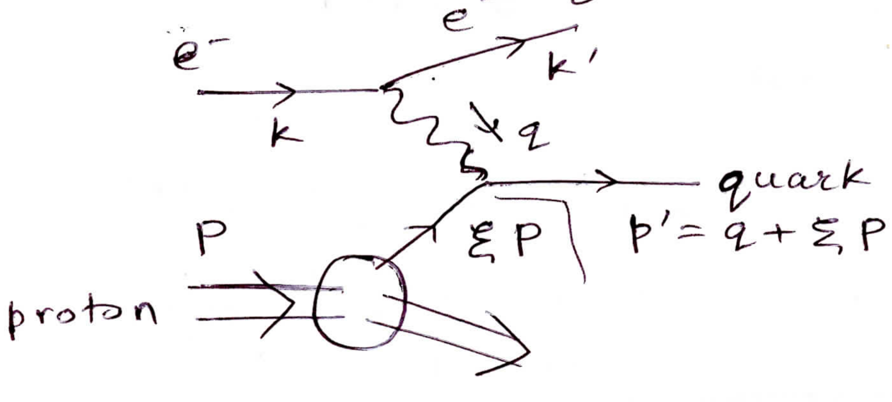
$$\sigma[e^-(k)\, \underset{\leftarrow \text{proton}}{p(P)} \to e^-(k') + \underset{\leftarrow \text{any hadronic state}}{X}]$$

$$= \int_0^1 d\xi \sum_f f_f(\xi)\, \sigma[e^-(k)\, q_f(\xi P) \to e^-(k') + q_f(p')]$$

$\sum_f$ over all constituents of proton (in general), but for QED (photon exchange) over quarks and antiquarks only (at leading order)

where $f_f(\xi)\, d\xi$ = probability of finding constituent f with longitudinal fraction ξ (f_f is parton distribution function or PDF)

— The corresponding "picture" is



(time →)

— Cross section for parton-level process is

$$\frac{d\sigma}{d\hat{t}}(e^- q \to e^- q) = \frac{2\pi\alpha^2 Q_f^2}{\hat{s}^2}\left[\frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2}\right]$$

borrowing result of electron-muon scattering: see

Peskin & Schroeder, section 5.4 or HW 10.2 from Phys 624

— Here Q_f is electric charge of quark and $\hat{s}, \hat{t}, \hat{u}$ are the Mandelstam variables (see Peskin & Schroeder, section 5.4 or Lahiri & Pal, section 7.6) for $2 \to 2$ parton-level process, i.e.,

$$\hat{s} = (k + \underbrace{\xi P}_{\text{quark momentum}})^2 \approx 2 \xi P \cdot k = \xi s$$

at proton-level

(neglecting all masses)

$$\hat{t} = (k' - k)^2 \equiv q^2 \equiv -Q^2$$

Using $\hat{s} + \hat{t} + \hat{u} = 0$, we have $\boxed{\hat{u} = Q^2 - \xi s} \Rightarrow$

$$\frac{d\sigma}{dQ^2} = \int_0^1 d\xi \sum_f f_f(\xi) Q_f^2 \frac{2\pi\alpha^2}{Q^4} \left[1 + \left(1 - \frac{Q^2}{\xi s} \right)^2 \right] \theta(\xi s - Q^2)$$

assume $Q^2 \gg (1 \text{ GeV})^2$

$\hat{u}$...

... upto $O[\alpha_s(Q^2)] \ll 1$ corrections from hard gluon exchange/emission

due to constraint: $\hat{s} \geq |\hat{t}|$

— Note that the $\hat{s}^2$ term in $[\hat{s}^2 + \hat{u}^2]$ factor (for parton-level cross-section comes from LH electron scattering from LH quark and RH ... RH ... whereas $\hat{u}^2$ is from LH electron scattering from RH quark and RH ... LH ... (obviously, we have summed/averaged over spins above)

— Finally, cross-section for $e^- \bar{q} \to e^- \bar{q}$ is same as for $e^- q \to e^- q$ (this fact has already been used in last formula)

— Next, we convert above formula into a more convenient / standard form

— First, we note that ξ is actually fixed by measurement of scattered electron momentum k' (and thus momentum transfer q):

mass of quark / parton

$$0 \simeq \underbrace{(\xi P + q)}_{\substack{\text{4-momentum} \\ \text{of scattered quark}}}^2 = 2\xi P.q + q^2 + \cancel{(\xi P)^2}_{0} \Rightarrow$$

$$\xi = \frac{Q^2}{2P.q} \equiv \boxed{x}$$ (clearly, x is directly measurable / observable)

— Express above formula for $d\sigma / dQ^2$ as doubly differential cross-section in x and Q^2: it is then a simple product of parton-level cross section and sum of PDF's evaluated at $\xi = x$ (cf. $\int d\xi$ in earlier formula)

— It is convenient to use dimensionless combination of kinematic variables: x and (instead of Q^2) $\boxed{y} \equiv \dfrac{2P.q}{2P.k} = \dfrac{2P.q}{s}$ ← momentum of proton

also directly measurable / observable

(physical picture: in proton rest frame, $y = \dfrac{q^0}{k^0}$, i.e., fraction of incident electron's energy that is transferred to hadronic system)

Rewriting y in terms of partonic variables (using

4-momentum of initial quark/parton $= \xi P$):

$$y = \frac{2(\xi P).(k-k')}{2(\xi P).k} = \frac{\hat{s}+\hat{u}}{\hat{s}}, \text{ i.e., } \frac{\hat{u}}{\hat{s}} = -(1-y)$$

– Kinematically allowed region: $0 \leq x, y \leq 1$

since it is a fraction ... use above formula(e)

– Finally, $Q^2 = xys$ and $d\xi dQ^2 = dx\, dQ^2 = \frac{dQ^2}{dy} dx\, dy$

$$= xs\, dx\, dy$$

(using above relations) to give

$$\frac{d\sigma}{dx\,dy}(e^- p \to e^- X) = \left(\sum_f x f_f(x) Q_f^2\right) \frac{2\pi\alpha^2 s}{Q^4}\left[1+(1-y)^2\right]$$

} again, observable ...

– Once photon propagator factor: $1/Q^4$, which involves both x & y, is removed, the dependence on x and y factorize: former is from PDF's, latter reflects helicities of interacting particles (see note at bottom of page 2): evidence that partons involved in deep inelastic scattering were fermions

– One can't determine separately the various PDF's from just electron scattering experiments ... but deep inelastic neutrino scattering does the job: formulae are similar to above, except that only LH particles (and RH antiparticles) ...