Chapter 7

Bulk Nuclear Properties and Nuclear Matter

The universe contains a remarkably wide variety of atomic nuclei, with mass numbers A (the sum of the numbers of proton Z and the neutron N) up to 250. While there are many interesting properties details that differentiate these nuclei from each other, there is also a powerful set of systematic trends and general properties that provide an important and useful framework for understanding the basic structure of nuclei. There properties are essentially determined by the so-called mean-field approach, in which one nucleon experiences a field which is the sum of the interaction with many other nucleons. This mean field property is a reflection that the density of the nucleons are relatively low and the interaction between the nucleons are relatively week. As such, the nucleon-nucleon correlation is small. The Hartree-Fock Mean field theory is main theoretical tool for dealing with systems with little correlations. Beyond that, the nucleon-nucleon correlations can be calculated using Bethe-Goldstone equations. These equations differ from free-space Schrodinger equation in that many-body Pauli blocking effects are taken into account. These theoretical tools are best illustrated in the example of nuclear matter in which the Coulomb potential is turned off and there are equal number of protons and neutrons. The finite size effect of the nuclear system will be taken up in the next Chapter.

7.1 Nuclear Radii and Densities

One of the first relevant properties of the nucleus was determined by Rutherford: the radius is only of order a few fm. The next major step was the use of electron scattering to accurately characterize the charge distribution of nucleons and nuclei. This technique was pioneered by Hofstadter in the 1950's. In Chapter 4, we considered the elastic scattering of electrons from nucleons. For spinless (i.e., $J = 0$) nuclei, elastic electron scattering is even simpler. The nuclear matrix element for this process is just $\langle 0|\hat{J}^\mu|0\rangle$, where J^μ is the electromagnetic current operator. For elastic scattering from a $J = 0$ nucleus, we have

$$\langle 0|\hat{J}^\mu|0\rangle = \delta^{\mu 0}\langle 0|\hat{\rho}|0\rangle \ , \tag{7.1}$$

so that only the charge operator contributes. The electron scattering cross section is then

$$\frac{d\sigma}{d\Omega} = \sigma_{\rm Mott} \cdot |F(\vec{q})|^2 \ ; \quad (J = 0 \to J = 0) \tag{7.2}$$

where $\sigma_{\rm Mott}$ is the cross section on a pointlike nucleus. An example of this cross section is shown in Figure 7.1.

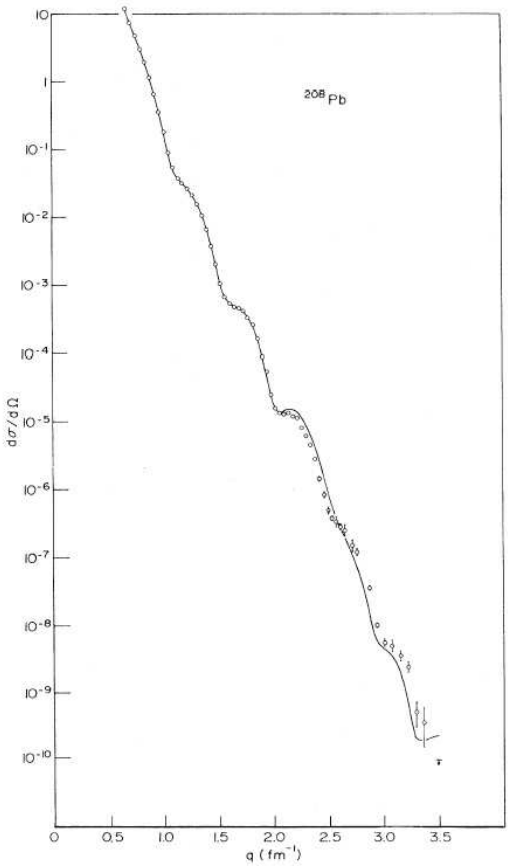

Figure 7.1: Cross section for elastic electron scattering from lead as a function of momentum transfer q along with a theoretical calculation.

The form factor $F(q)$ is directly related to the Fourier transform of the nuclear charge distribution. First define

$$\rho(q) \equiv \int \langle \rho(x) \rangle e^{-i\vec{q}\cdot\vec{x}} d^3\vec{x}\,, \tag{7.3}$$

where for a spin 0 nucleus $\langle \rho(x) \rangle$ is spherically symmetric. Then use the identity

$$e^{-i\vec{q}\cdot\vec{x}} = \sum_{l=0}^{\infty} (2l+1) i^l j_l(qr) P_l(\cos\theta) \tag{7.4}$$

and the definition

$$j_0(qr) = \frac{\sin qr}{qr}\,. \tag{7.5}$$

Only the $l = 0$ term survives due to the spherical symmetry of $\langle \rho(x) \rangle$

$$\rho(q) = \frac{4\pi}{q} \int_0^{\infty} \rho(r) \sin(qr) r dr\,. \tag{7.6}$$

We can then expand $\sin(qr)$ and obtain the relation:

$$\rho(q) = \frac{4\pi}{q}\int \rho(r)\left[qr - \frac{1}{6}q^3r^3 + ...\right] rdr \tag{7.7}$$

$$= 4\pi\int \rho(r)r^2dr - \frac{1}{6}q^2\int r^2\rho(r)4\pi r^2dr + ... \tag{7.8}$$

$$= Ze(1 - \frac{1}{6}q^2\langle r^2\rangle + ...) \tag{7.9}$$

We should note two important properties of $\rho(q)$:

$$\lim_{q\to 0}\rho(q) = Ze \qquad \text{(total charge)} \tag{7.10}$$

$$\lim_{q^2\to 0}\frac{dF}{dq^2} = \frac{1}{Ze}\lim_{q^2\to 0}\frac{d\rho}{dq^2} = -\frac{1}{6}\langle r^2\rangle \tag{7.11}$$

or

$$\langle r^2\rangle = -6\frac{dF}{dq^2}\Big|_{q^2=0} \qquad \text{(mean square charge radius)} \tag{7.12}$$

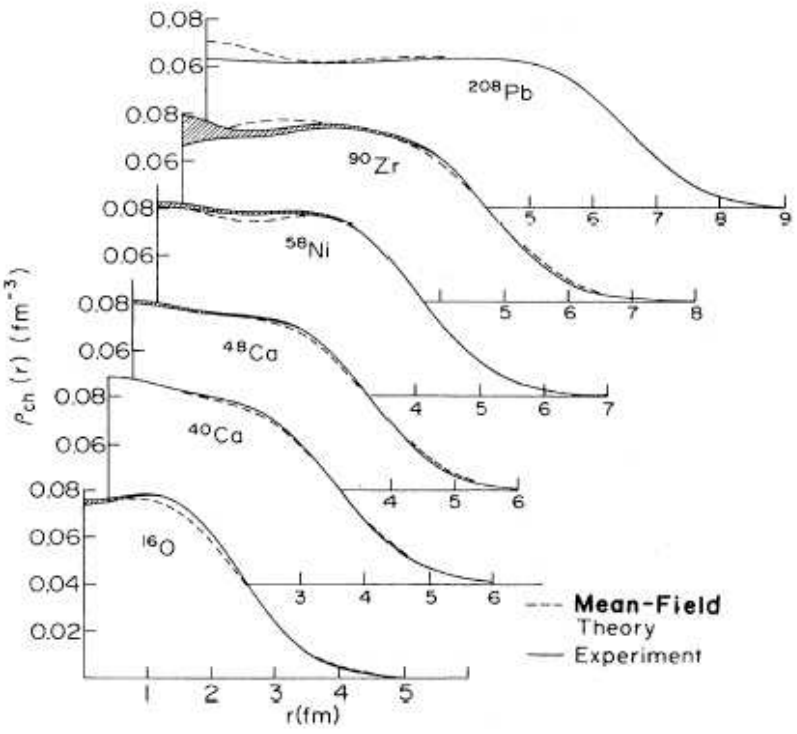

Figure 7.2: Charge density distributions for nuclei as determined from elastic electron scattering along with theoretical calculations.

The general strategy is to measure $\rho(q)$ and Fourier transform to obtain $\rho(r)$. As shown in Figure 7.2, one finds a nuclear charge distribution that is well described by a Wood-Saxon form

$$\rho(r) = \frac{\rho_0}{1 + e^{(r-R)/a}} \, . \tag{7.13}$$

The basic properties of this distribution are the following.

- (a)$\rho_0 \sim 0.08\, e\text{fm}^{-3}$
- (b)$c = r_0 A^{\frac{1}{3}}$; $r_0 \cong 1.2$fm.

Therefore, as one adds nucleons the nuclear volume simply grows with the number of nucleons in such a way that the density of nucleons (per unit volume) is constant. That is, nuclei seem to behave like an incompressible fluid of constant density. This property is very different than that of atoms.

The elastic electron scattering probes the charge distribution of the protons. The neutron distribution in the nucleus can be very different in principle. However, if one assumes that the strong interactions dominate the nuclear structure and the isospin symmetry is good, the neutron and proton distributions shall not be too different. The neutron distribution in a nucleus can be measured in principle by parity-violating electron scattering with Z-boson exchanges, because the neutral current charge of the proton almost vanishes.

7.2 Fermi Gas Model

The fact that the nucleon density is approximately constant throughout the nuclear interior implies that it is not energetically more favorable to be in one location compared to any other. Thus, we expect that the mean potential will be approximately constant throughout the nuclear volume. We may then approximate the nucleus as a free Fermi gas (i.e., non-interacting fermions) confined to a well which approximates the nuclear volume. The kinetic energies associated with localization to the nuclear volume ($\sim$ few MeV) are large compared to room temperature $T_{\text{room}} = \frac{1}{40}$ eV, so it is safe to use a degenerate Fermi gas ($T = 0$) for temperatures below about $\sim 10^9$ °K. (Such large temperatures are in fact encountered in stellar environments.

We are allowed to place 4 particles (spin-isospin) in each orbital. The density of available states is then given by the expression

$$dN = 4 \times \frac{d^3p}{(2\pi\hbar)^3}\Omega \tag{7.14}$$

$$= 4 \times \frac{4\pi p^2 dp}{(2\pi\hbar)^3}\Omega \tag{7.15}$$

where Ω is the nuclear volume. If we integrate to the highest filled orbital at momentum p_F, we then count the total number of nucleons

$$A = \frac{16\pi\Omega}{8\pi^3\hbar^3}\int_0^{p_F} p^2 dp = \frac{2\Omega}{3\pi^2\hbar^3}p_F^3 = \frac{2\Omega}{3\pi^2}k_F^3 \, . \tag{7.16}$$

Then the nuclear density is

$$\rho = \frac{A}{\Omega} = \frac{2}{3\pi^2}k_F^3 \, . \tag{7.17}$$

We can compute the average kinetic energy of the nucleons as follows.

$$\langle T\rangle = \frac{1}{2M_N}\frac{\int_0^{p_F} p^4 dp}{\int_0^{p_F} p^2 dp} = \frac{1}{2M_N}\frac{p_F^5/5}{p_F^3/3} \tag{7.18}$$

$$= \frac{3}{5}\frac{p_F^2}{2M_N} \tag{7.19}$$

We can numerically determine the Fermi momentum p_F by using the known value of nuclear density (determined from electron scattering charge densities). Recall that the nuclear radius is given by $R = 1.2\,\mathrm{fm}\ A^{1/3}$, which implies

$$\Omega = \frac{4\pi}{3}(1.2)^3 A\,. \tag{7.20}$$

Thus we may evaluate the nuclear density as

$$\rho = \frac{3}{4\pi(1.2)^3}\ \mathrm{fm}^{-3} \tag{7.21}$$

$$= .14\ \mathrm{fm}^{-3} \tag{7.22}$$

$$= 2.34\times 10^{14}\ \mathrm{g/cm}^3\,. \tag{7.23}$$

We then use

$$k_F = \left(\frac{3\pi^2\rho}{2}\right)^{1/3}\ \mathrm{fm}^{-1} \tag{7.24}$$

to obtain

$$p_F = \hbar k_f = 250\ \mathrm{MeV/c}\,. \tag{7.25}$$

(This value agrees very well with the analysis of $F(p_z)$ distributions determined from quasielastic electron scattering measurements!) We can now numerically evaluate the average kinetic energy as

$$\langle T\rangle = \frac{3}{5}\frac{(\hbar k_f)^2}{2M_n} = \frac{3}{5}\frac{(197.3\times 1.3)^2}{2\times 938} \tag{7.26}$$

$$= 21\ \mathrm{MeV} \tag{7.27}$$

and the maximum kinetic energy is then

$$T_{max} = \frac{5}{3}\langle T\rangle = 35\ \mathrm{MeV}. \tag{7.28}$$

We note that, under laboratory conditions, the zero temperature limit is certainly valid.

Clearly the depth of the well has to be larger than 35 MeV. If a nucleon has binding energy of order 15 MeV or so, as in nuclear matter, the depth of the well will be about $15 + 35 = 50$ MeV.

7.3 Independent Particles and Quasi-Elastic Electron Scattering

In probing the nuclear structure with electromagnetic probes, the spatial resolution of the virtual photon is governed by the momentum transfer $\vec{q}$. For momentum transfers up to several hundred MeV, elastic and inelastic scattering from the nucleus can be used to study properties of the ground

state (such as the charge distribution) and low energy excitations. At higher momentum transfers up to 1-2 GeV/c, the dominant process involves the interaction of the virtual photon with an individual nucleon in the nucleus. (The amplitude for coherent interaction with several or many nucleons decreases rapidly in magnitude at these higher momentum transfers.) The simplest process of this type involves the quasifree knockout of a nucleon, where the struck nucleon is not internally excited but rather recoils elastically. This process is known as "quasielastic" scattering and is a widely used tool for the study of the properties of the nucleon in the nuclear medium.

The major effect of the nuclear medium on the struck nucleon is the momentum associated with the localization of the initial state. Thus we consider the case of electron scattering from an initial state nucleon which is in motion in the laboratory frame. Let the initial nucleon momentum be given by $\vec{p} = (p_z, p_\perp)$ where

$$p_z \equiv \frac{\vec{p}\cdot\vec{q}}{|\vec{q}|} \,. \tag{7.29}$$

Then we still have $-q^2 = 2p\cdot q$, since this is a Lorentz invariant statement, so

$$-q^2 = 2p\cdot q = 2m\nu - 2p_z|\vec{q}| \tag{7.30}$$

where we have ignored the (rather small) effect of the kinetic energy of the initial nucleon and set its energy equal to its mass (i.e., $E = m$). Thus, the electron energy loss ν is shifted by

$$\begin{aligned} \nu &= -\frac{q^2}{2m} + \frac{|\vec{q}|}{m}p_z & (7.31)\\ &= \nu_0 + \frac{|\vec{q}|}{m}p_z\left(1 + \frac{2k}{m}\sin^2\frac{\theta}{2}\right)^{-1} & (7.32)\\ &\cong \nu_0 + \frac{|\vec{q}|}{m}p_z \quad . & (7.33) \end{aligned}$$

So to lowest order in p_z we have the relation $\frac{d\nu}{dp_z} = \frac{|\vec{q}|}{m}$ or $\frac{dp_z}{d\nu} = \frac{m}{|\vec{q}|}$. Now let's write the elastic cross section

$$\left.\frac{d^2\sigma}{d\Omega d\nu}\right|_{el} = \left.\frac{d\sigma}{d\Omega}\right|_{el} \cdot \underbrace{\delta(\nu + q^2/2m)}_{\frac{dP}{d\nu}=\text{probability/unit }\nu} \quad . \tag{7.34}$$

Then we can generalize this expression to the quasielastic cross section

$$\left.\frac{d^2\sigma}{d\Omega d\nu}\right|_{QE} = \left.\frac{d\sigma}{d\Omega}\right|_{el} \times \underbrace{\frac{dP(p_z)}{dp_z}\frac{dp_z}{d\nu}}_{\frac{dP}{d\nu}} \tag{7.35}$$

Let's assume the nucleus (spherically symmetric) has a momentum distribution $n(p)$:

$$\int d^3p\, n(p) = 4\pi\int_0^\infty p^2 dp\, n(p) = 1 \,. \tag{7.36}$$

Now $p^2 = p_z^2 + p_\perp^2$ and for fixed p_z we have $2p\,dp = 2p_\perp\, dp_\perp$ and

$$\begin{aligned} \frac{dP}{dp_z} &= \int_0^\infty n(p_z, p_\perp) d^2p_\perp = 2\pi\int_0^\infty n(p_z, p_\perp)\, p_\perp\, dp_\perp & (7.37)\\ &= 2\pi\int_{p_z}^\infty n(p)\, p\, dp \equiv F(p_z) \,. & (7.38) \end{aligned}$$

If we then recall that all of the nucleons in a nucleus are equivalent (they are identical fermions and the wave function is totally antisymmetrized) then we can sum over all the nucleons and use $\frac{dp_z}{d\nu} = \frac{m}{|\vec{q}|}$ to obtain the quasielastic cross section formula:

$$\left.\frac{d\sigma}{d\Omega d\nu}\right|_{QE} = \left.\frac{d\bar{\sigma}}{d\Omega}\right|_{el} \cdot \frac{m}{|\vec{q}|} \underbrace{F(p_z)}_{\text{measure!!}} \tag{7.39}$$

where we have defined the effective nucleon elastic cross section as

$$\left.\frac{d\bar{\sigma}}{d\Omega}\right|_{el} \equiv Z \cdot \left.\frac{d\sigma}{d\Omega}\right|_{\text{proton}} + N \left.\frac{d\sigma}{d\Omega}\right|_{\text{neutron}} . \tag{7.40}$$

Thus one can measure the ν dependence of the quasielastic cross section and use the known nucleon elastic cross sections to determine the "longitudinal momentum distribution" of the nucleons $F(p_z)$. This is a basic property of the nucleon orbitals in the nucleus.

Figure 7.3 shows a sample of quasielastic electron scattering data for a variety of nuclei analyzed for the longitudinal momentum distribution $F(p_z)$. The extraction of $F(p_z)$ for a variety of nuclei for $2 < A < 200$ indicates a remarkable property. For all nuclei with $A > 12$ one finds a typical mean momentum for the initial nucleon of about 150 MeV.

This is consistent with the view that these nuclei all have about the same average density as determined from elastic electron scattering. A very simple model assumes that the nucleons are a free Fermi gas confined to a spherical volume of radius R. Then the density of nucleons is related to the Fermi momentum p_F, and a constant p_F is indicative of a constant nucleon density. In fact, the momentum distribution of a free Fermi gas is quite simple and can be used to obtain an analytic expression for the quasielastic scattering cross section using the above formulae.

More detailed information about the initial nucleon in quasielastic electron scattering can be obtained by measuring the momentum of the struck nucleon in the final state. (This is most practical for protons, but neutron experiments are becoming feasible also.) The recoiling nucleon has some probability to further interact with the residual nucleus, but one can account for this effect by using "distorted" waves for the final nucleon. In any case, the assumption that there is no final state interaction is not unreasonable at larger recoil momenta (> 500 MeV/c) and we will proceed using this ansatz.

The determination of the final momentum of the nucleon allows us to solve uniquely (using energy and momentum conservation) for the initial nucleon momentum and energy. The recoiling residual nucleus $A - 1$ has very little kinetic energy so we neglect it here (one can include it in a more detailed analysis).

$$\vec{p} = \vec{p}' - \vec{q} \tag{7.41}$$

$$E = E' - \nu \tag{7.42}$$

The initial energy E contains the binding energy of the struck nucleon. Therefore, one can select the initial binding energy of the nucleon by selecting scattering events with the desired value of E.

An example of data from a quasielastic $(e, e'p)$ experiment is shown in Figure 7.4. The distribution of binding energies shows a great deal of structure with many peaks associated with specific bound state orbits. Each peak has a "momentum distribution" corresponding to the squared

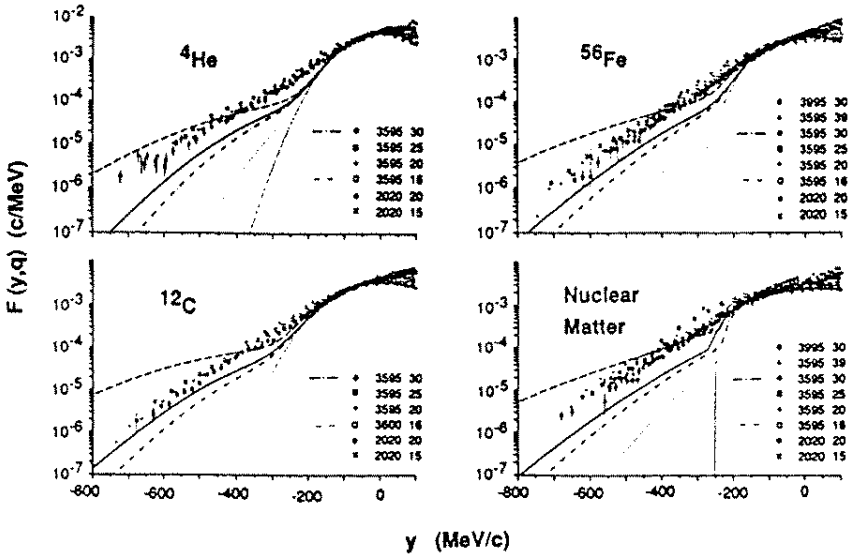

Fig. 17. Comparison between experimental and theoretical scaling functions. Dotted line for the one-body contribution (for finite nuclei divided by 0.8), double-dotted and dash-dotted lines including many-body contributions, solid line for the asymptotic scaling function and dashed line for the longitudinal momentum distribution (from ref. [136]).

Figure 7.3: Quasielastic electron scattering from several nuclei showing the universal nature of the peak corresponding to single nucleon knockout kinematics, and the scaling behavior as a function of q^2.

momentum-space wave function for that particular bound state orbital. Analysis of these distributions leads to the inescapable and remarkable conclusion that the bound nucleons are well-described by motion in a potential well the size of the *whole* nucleus. (The Fourier transform of the momentum-space wave function is in fact the coordinate-space wave function that corresponds to the spatial distribution for that bound nucleon.)

The results from such recent $(e, e'p)$ experiments support many decades of phenomenological development of nuclear theory based on the assumption that nucleons move freely throughout the nuclear volume in a "mean potential" generated by the average effect of all the other $A-1$ nucleons. The validity of this picture is, at first glance, rather surprising given the short-range nature of the force between nucleons which might seem to favor a view where the nucleons "rattle" around by bouncing off each other in distances short compared with the nuclear radius. However, quantum mechanically confining the nucleons to such smaller effective volumes would entail a large increase in the kinetic energy of the nuclear system. Therefore, the lowest energy state is one with larger spatial orbitals generated by an effective average potential. The theoretical description of dense strongly interacting matter as quasifree particles in a potential well is a very interesting tale in quantum many-body physics which we will study in some more detail later.

We have already discussed how the nuclear force between nucleons saturates very quickly due

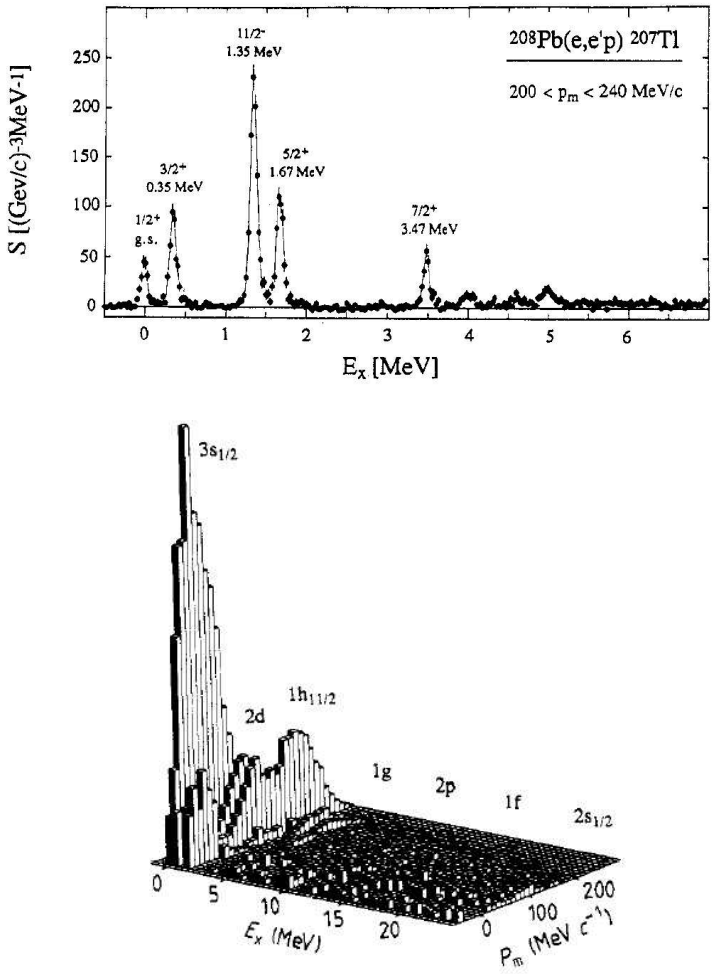

Figure 7.4: Cross section data for the $^{208}\mathrm{Pb}(e,e'p)^{207}\mathrm{Tl}^*$ reaction displaying the orbitals of the nucleon for the different final states.

to the short-range nature of the force. This leads to the approximately constant binding energy per nucleon observed for all but the very lightest nuclei. We have also seen that there is considerable evidence that the nucleons occupy orbitals that fill the complete nuclear volume even though the range of the nuclear force is smaller than the nuclear radius. This can be understood as due to the Pauli-blocking of already filled states. Since there are no empty orbitals at low energies, there is no opportunity to scatter and change orbitals. Thus we are led to the picture that the nucleons can move under the influence of a mean potential generated by the collective effect of all the other nucleons. In addition, the nucleons predominantly occupy orbitals that are the eigenstates of such a mean potential.

It is possible to construct the mean potential from the nucleon- nucleon potential using quantum many-body theory, and we will study this later. However, we will find it instructive to first adopt a very simple mean potential in order to explore the consequences of this picture. We will then be able

to add some corrections and develop a very successful formula for the binding energy systematics of nuclei.

7.4 Nuclear Binding Energy and Bethe-Weizsäcker formula

Another remarkable property of atomic nuclei (also very different from atoms) is that the binding energy per nucleon is approximately constant at 8 MeV per nucleon (Figure 7.5. If one adds a nucleon to the nucleus and it interacts with all the other nucleons simutaneously, then the total binding energy would grow as A^2, and the binding energy per nucleon would grow in proportion to A. The fact that the binding energy per nucleon is constant is an indication that only nearest neighbor interactions are significant. Each nucleon that is added to the nucleus sees binding effects from its nearest neighbors only, which contribute some fixed amount (in this case 8 MeV). Therefore, we conclude that nucleons only interact with their nearest neighbors and, from the constant nuclear density, this nearest neighbor distance is only of order 1 fm.

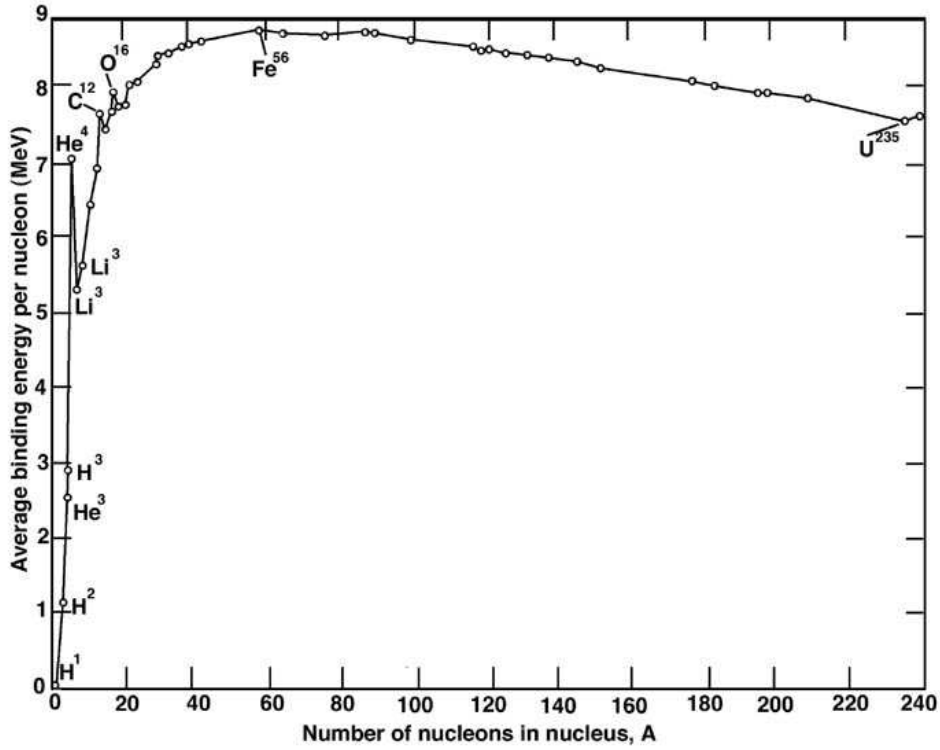

Figure 7.5: Binding energy per nucleon for stable nuclei as a function of nuclear mass number A. The binding energy saturates at the value $B/A \sim 8$ MeV/nucleon. The most stable nucleus is ^{56}Fe.

Next we consider corrections to the Fermi gas model developed above. These corrections account for the potential energy of the nucleons, the surface of a finite-sized nucleus, the Coulomb energy associated with the protons, the implications of isospin symmetry, and the tendency for nucleons to form pairs with $J = 0$. These considerations will lead to the Bethe-Weizsäcker formula for the

binding energy of a finite nucleus of A nucleons and atomic number Z

$$B(A,Z) = a_V A - a_S A^{2/3} - a_C \frac{Z(Z-1)}{A^{1/3}} - a_A \frac{(N-Z)^2}{A} + \Delta E_{\rm pair} \,. \tag{7.43}$$

The "pairing energy" is given by

$$\Delta E_{\rm pair} = \delta \cdot \frac{a_P}{A^{1/2}}; \quad \delta = \begin{cases} 1 & \text{even} - \text{even} \\ 0 & \text{even} - \text{odd} \\ -1 & \text{odd} - \text{odd} \end{cases} \tag{7.44}$$

and the five terms with their empirically determined constants are

Volume term:	$a_V = 15.85$ MeV
Surface term:	$a_S = 18.34$ MeV
Coulomb term:	$a_C = 0.71$ MeV
Symmetry term:	$a_A = 23.21$ MeV
Pairing term:	$a_P = 12$ MeV

Table 7.1: Parameters in Bethe-Weizsäcker formula.

Volume Energy

This term arises from both the kinetic and potential energy associated with bulk volume of the nuclear system

$$\langle T + V \rangle = \langle T \rangle + \langle V \rangle \,. \tag{7.45}$$

The kinetic energy is that of the free Fermi gas that we have already considered:

$$T = A \cdot \langle T \rangle = \frac{3}{5} \frac{p_F^2}{2M_N} A = \frac{3}{5} \frac{\hbar^2}{2M_N} \left(\frac{3\pi^2 \rho}{2} \right)^{2/3} A \,. \tag{7.46}$$

For the potential energy, we consider only a central potential V_C between nucleons (other potential energy contibutions associated with the spins will tend to yield an average central potential when one integrates over all directions).

$$V = \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} \int \rho(\vec{r}_i) \rho(\vec{r}_j) V_C(r_{ij}) d^3 r_i d^3 r_j \tag{7.47}$$

$$= \frac{A(A-1)}{2} \int \rho(\vec{r}_1) \rho(\vec{r}_2) V_C(r_{12}) d^3 r_1 d^3 r_2 \tag{7.48}$$

where we have used the fact that all nucleons are equivalent (the total wave function is antisymmetrized) and multiplied the potential energy of two nucleons by the number of pairs $A(A-1)/2$. The nucleon density will be considered to be a constant over the nuclear volume Ω

$$\rho(\vec{r}_i) \equiv \text{prob/vol to find the i}th \text{ particle at } \vec{r}_i \tag{7.49}$$

$$\cong \frac{1}{\Omega} \,. \tag{7.50}$$

Since V_C is short-ranged and the nuclear medium is of uniform density, the integral of V_C over the position of nucleon 2 is a constant independent of the location of nucleon 1:

$$\int V_C(r_{12})\rho(\vec{r}_2)d^3\vec{r}_2 \cong \frac{1}{\Omega}\int V_C(\vec{r})d^3r \tag{7.51}$$

$$\equiv \frac{\bar{V}_C}{\Omega} \tag{7.52}$$

Then the total potential energy can be written as

$$V \cong \frac{A^2}{2\Omega}\bar{V}_C \tag{7.53}$$

$$= \frac{1}{2}A\rho\bar{V}_C \quad (< 0 \text{ for binding with attractive forces}). \tag{7.54}$$

Therefore, we have

$$a_V = -\frac{T+V}{A} \cong c_2\rho - c_1\rho^{2/3} \tag{7.55}$$

with positive constants c_1 and c_2 associated with the kinetic energy and potential energy contributions, respectively.

From such a simple relation, we would predict a rather surprising property of nuclear matter. At large enough ρ, the c_2 term dominates a_V, and then a_V increases as ρ increases. Thus, if the nuclear density becomes high enough it will be energetically favorable for the nucleus to increase its density without limit. Thus we are led to the unfortunate conclusion that nuclei should collapse. Since this obviously does not occur in nature, there must be some additional effect. It turns out that the nuclear force actually becomes strongly repulsive at short distances. This implies that c_2 cannot be assumed to be constant, but in fact decreases and becomes negative at high ρ. This repulsive core of the nuclear interaction is then responsible for stabilizing nuclear matter at finite density.

Surface Energy

There can be 2 contributions to the surface term. One is the finite volume effect on the density of states ("finite" Fermi gas) which affects the kinetic energy. The other is due to the finite volume effect on the potential.

We first consider the surface correction to the kinetic energy in the Fermi gas. For calculational convenience we will use a cubical box for our Fermi gas. (The shape of the volume will only change the result by a simple geometric factor, but the general form of our result will be correct for any simple shape.) Note that the integral we previously used for the density of states included states with $k_x = 0$, $k_y = 0$, and $k_z = 0$. For a cube, these are not allowed since the wave functions of these orbitals

$$\psi = \sin(k_x s)\sin(k_y y)\sin(k_z z) \tag{7.56}$$

will vanish. Now let's count the number of states with $k_x = 0$:

$$dN_x = 4 \times \frac{L^2 dk_y dk_z}{(2\pi)^2} = \frac{4S\,2\pi k dk}{6(2\pi)^2} \tag{7.57}$$

$$= \frac{S}{3\pi}kdk \tag{7.58}$$

where $S = 6L^2$ is the total surface area of the cubical volume. Therefore the total number of states should be modified by

$$dN = \left(\frac{2\Omega}{\pi^2}k^2 - \frac{S}{\pi}k\right)dk \tag{7.59}$$

where we have subtracted three times dN_x to account for $k_y = 0$ and $k_z = 0$ states also. Then we repeat the Fermi gas calculations with this correction included:

$$\begin{aligned}
A &= \int_0^{k_F}\left(\frac{2\Omega}{\pi^2}k^2 - \frac{S}{\pi}k\right)dk && (7.60)\\
&= \frac{2\Omega}{3\pi^2}k_F^3 - \frac{S}{2\pi}k_F^2 && (7.61)\\
&= \frac{2\Omega}{3\pi^2}k_F^3\left(1 - \frac{3\pi}{4}\frac{S}{\Omega}\frac{1}{k_F}\right) && (7.62)\\
\langle T\rangle &= \frac{\hbar^2}{2M_N}\frac{\int_0^{k_F}\left(\frac{2\Omega}{\pi^2}k^4 - \frac{S}{\pi}k^3\right)dk}{A} && (7.63)\\
&= \frac{\hbar^2}{2M_N}\left[\frac{2\Omega}{5\pi^2}k_F^5 - \frac{S}{4\pi}k_F^4\right]\Big/A && (7.64)\\
&= \frac{\hbar^2}{2M_N}\left(\frac{2\Omega}{5\pi^2}k_F^5\right)\left[1 - \frac{5\pi S}{8\Omega}\frac{1}{k_F}\right]\Big/A && (7.65)\\
&\cong \frac{3}{5}\frac{\hbar^2k_F^2}{2M_N}\left[1 + \frac{\pi}{8}\frac{S}{\Omega}\frac{1}{k_F} + \cdots\right] && (7.66)
\end{aligned}$$

We now use the approximate expression for the ratio of surface area to volume:

$$\frac{S}{\Omega} \cong \frac{4\pi r_0^2 A^{2/3}}{\frac{4\pi}{3}r_0^3 A} \cong \frac{3}{r_0 A^{1/3}} \tag{7.67}$$

and obtain

$$\langle T\rangle = \frac{3}{5}E_F + \frac{9}{40}E_F\frac{\pi}{r_0k_F}\cdot\frac{1}{A^{1/3}}\,. \tag{7.68}$$

We therefore find the following value for the kinetic energy part of a_S:

$$a_S(T) = \frac{9}{40}E_F\frac{\pi}{r_0k_F} \approx 18\text{ MeV}. \tag{7.69}$$

We should remember that we expect this to be valid only to within a geometric factor like 2 or 3.

For the potential energy term we can use a simple geometric analysis. The volume associated with the nuclear surface is a shell of thickness approximately equal to the range of the nuclear force r_1

$$d\Omega \cong 4\pi R^2 r_1\,. \tag{7.70}$$

Then the potential energy is reduced as follows:

$$\begin{aligned}
V &\cong \frac{1}{2}A\rho\bar{V}_C\left(1 - \frac{d\Omega}{\Omega}\right) && (7.71)\\
&\cong \frac{1}{2}A\rho\bar{V}_C\left(1 - \frac{3r_1}{R}\right) && (7.72)\\
&\cong \frac{1}{2}A\rho\bar{V}_C - \frac{3}{2}\rho\frac{r_1}{r_0}A^{2/3}\bar{V}_C && (7.73)
\end{aligned}$$

where we have used $R = r_0 A^{1/3}$ for the nuclear radius. Therefore, $a_S(V) \approx -\frac{3}{2}\rho\frac{r_1}{r_0}\bar{V}_C$ (>0 for an attractive force).

For a simple square well potential it is easy to show that

$$V = \frac{1}{2}A\rho\bar{V}_C\left[1 - \frac{9}{16}\frac{r_1}{R} + \frac{1}{32}\left(\frac{r_1}{R}\right)^3\right]. \tag{7.74}$$

Coulomb Energy

The charge density is given by

$$\rho_q = \frac{Z}{A}\rho e = \frac{Ze}{\Omega} \tag{7.75}$$

which then yields the following for the Coulomb energy

$$V_c = \frac{1}{2}\int \rho_q^2 \frac{d^3r_1 d^3r_2}{r_{12}} \tag{7.76}$$

$$= \rho_q^2 \int_0^R \frac{4\pi r^3}{3}\frac{4\pi r^2 dr}{r} \quad \text{(Energy to assemble charge distribution)} \tag{7.77}$$

$$= \frac{4\pi}{3}\rho_q^2 \cdot 4\pi\frac{R^5}{5} \tag{7.78}$$

$$= \frac{3}{5}\frac{Z^2e^2}{R}$$

$$= \tfrac{3}{5}\tfrac{Z^2e^2}{r_0}A^{-1/3}. \tag{7.79}$$

Thus we find the expression for a_C

$$a_C = \frac{3}{5}\frac{e^2}{r_0} \approx 0.71 \text{ MeV}. \tag{7.80}$$

Symmetry Energy

The nuclear force prefers $T = 0$, so nuclei with $Z = N$ are expected to have extra stability and so maximize the binding energy B. Let's assume two Fermi gases consisting of Z protons and N neutrons. We will allow the relative number of protons and neutrons to vary (i.e., $Z \neq N$), but we will keep $A = Z + N$ fixed. Now for the two Fermi gases we repeat the analysis as follows. First we define the densities of the two components

$$\rho_0 = \frac{A}{\Omega};\ \rho_p = \frac{Z}{A}\rho_0 \equiv x\rho_0;\ \rho_n = \frac{N}{A}\rho_0 = (1-x)\rho_0\,. \tag{7.81}$$

Then we write the total kinetic energy for symmetric ($Z = N$) matter as

$$T_0 = A \cdot \frac{3}{5}\frac{\hbar^2}{2M_N}(3\pi^2\rho_0/2)^{2/3}\,. \tag{7.82}$$

For $Z \neq N$ we have the following

$$T_p = Z \cdot \frac{3}{5}\frac{\hbar^2}{2M_N}(3\pi^2\rho_p)^{2/3} = A \cdot \frac{3}{5}\frac{\hbar^2}{2M_N}(3\pi^2\rho_0)^{2/3}x^{5/3} \tag{7.83}$$

$$T_n = A \cdot \frac{3}{5}\frac{\hbar^2}{2M_N}(3\pi^2\rho_0)^{2/3}(1-x)^{5/3}\,. \tag{7.84}$$

The total kinetic energy is the sum of the two components

$$T_p + T_n = 2^{2/3} T_0 [x^{5/3} + (1-x)^{5/3}] \,. \tag{7.85}$$

We then expand about $x = \frac{1}{2}$: $\delta x \equiv x - \frac{1}{2} = \frac{Z-N}{2A}$.

$$x^{5/3} = \frac{1}{2^{5/3}} + \frac{5}{3}\frac{1}{2^{2/3}}\delta x + \frac{10}{9}2^{1/3}\frac{\delta x^2}{2} + \cdots \tag{7.86}$$

$$(1-x)^{5/3} = \frac{1}{2^{5/3}} - \frac{5}{3}\frac{1}{2^{2/3}}\delta x + \frac{10}{9}2^{1/3}\frac{\delta x^2}{2} + \cdots \tag{7.87}$$

The kinetic energy is then given by

$$\begin{aligned} T_p + T_n &= 2^{2/3} T_0 \left[\frac{2}{2^{5/3}} + \frac{20}{9} \cdot \frac{1}{2^{2/3}} \delta x^2 \right] \\ &= T_0 + \tfrac{5}{9} T_0 \left(\tfrac{(Z-N)}{A} \right)^2 \qquad (7.88) \\ &= T_0 + a_A \frac{(N-Z)^2}{A} \qquad (7.89) \end{aligned}$$

Thus we conclude that

$$a_A = \frac{5}{9}\frac{T_0}{A} = \frac{5}{9} \times 21 \text{ MeV} = 11.5 \text{ MeV}. \tag{7.90}$$

There should also be some contribution from potential energy, but clearly this effect associated with the asymmetry of Fermi energies already qualitatively explains the empirical value of a_A.

Pairing Energy

The pairing energy takes into account the tendency of like nucleons to form pairs (rather like Cooper pairs in a superconductor) in order to lower the energy of the nuclear system. Thus if either the neutron number N or proton number Z is even there is some additional binding of $12/\sqrt{A}$ MeV relative to the case where both N and Z are odd integers. If both N and Z are even, then one gets twice this value in additional binding energy.

Summary

Note that for a given A, $E_B(A,\ Z)$ is quadratic in Z. The Z which maximizes $E_B(A,\ Z)$ is determined by a combination of the Coulomb and symmetry energies. For small A, the symmetry term dominates so N=Z is the maximum. At higher $A > 40$, the Coulomb term becomes important and the maximum occurs at $Z < N$. This explains the trend of the "valley" of stable nuclei as a function of Z and N. Typically, we find the pattern of energies shown in Fig. 7.6 for the cases of even and odd A. Note that for even A there are 2 parabolas for even-even and odd-odd nuclei separated by the pairing energy, while for odd A there is only one parabola.

Many properties of finite nuclei can be related to the simpler system of infinite nuclear matter. The matter is assumed to consist of equal numbers of protons and neutrons, and we turn off the Coulomb interaction. The formalism can also be modified and applied to finite nuclei.

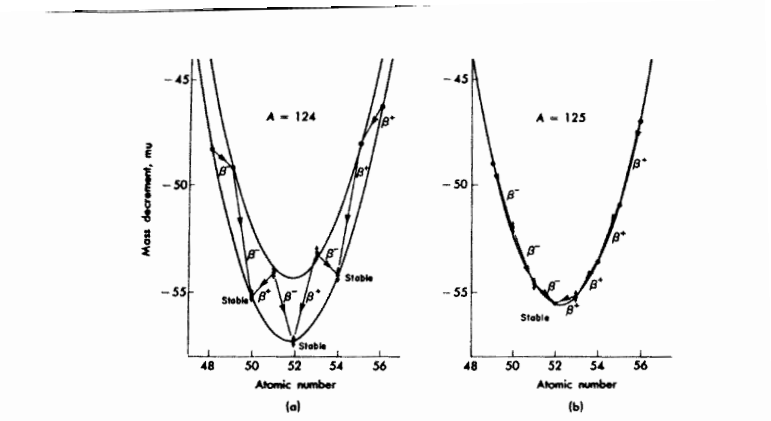

Fig. 6-2 (a) Mass excesses for $A = 124$; the lower curve is for even-even, and the upper one for odd-odd, nuclides. (b) Mass excesses for $A = 125$.

Figure 7.6: Ground state energies for $A = 124$ (left) and $A = 125$ (right).

7.5 Mean Field Theory and Independent Particle Approximation

As a first approximation, nuclei and nuclear matter are well-represented by independent particles (nucleons) moving in a mean potential. The best such description is obtained by utilizing the Hartree-Fock formalism.

The A-particle Hamiltonian is

$$\hat{H} = \sum_i \frac{p_i^2}{2M} + \frac{1}{2}\sum_{i\neq j} V(\vec{r}_i,\ \vec{r}_j) \tag{7.91}$$

$$\equiv \sum_i T_i + \frac{1}{2}\sum_{i\neq j} V_{ij} \tag{7.92}$$

where the N-N potential has the symmetry

$$V(\vec{r}_i,\ \vec{r}_j) = V(\vec{r}_j,\ \vec{r}_i) \rightarrow V_{ij} = V_{ji}\,. \tag{7.93}$$

$V(\vec{r}_i,\ \vec{r}_j)$ may also depend on spin-isospin but we will omit such a dependence and assume a central potential for simplicity.

Now choose an (in principle) arbitrary single particle Hamitonian $\hat{h}(\vec{r}_i,\ \vec{p}_i) \equiv \hat{h}_i$, with eigenvectors $\{\phi_k(\vec{r}_i)\}$ and eigenvalues E_k:

$$\hat{h}_i\phi_k(\vec{r}_i) = E_k\phi_k(\vec{r}_i). \tag{7.94}$$

The lowest energy (ground state) A-particle solution for the Hamiltonian $\hat{H}_0 = \sum_{i=1}^{A} h_i$ is the Slater determinant

$$\psi(1,2\ldots A) = \frac{1}{\sqrt{A}}\begin{vmatrix} \phi_\alpha(1) & \phi_\alpha(2) & \ldots & \phi_\alpha(A) \\ \phi_\beta(1) & \phi_\beta(2) & \ldots & \phi_\beta(A) \\ \vdots & \vdots & & \vdots \end{vmatrix} \tag{7.95}$$

where $\{E_\alpha,\ E_\beta, \ldots\}$ are lowest A energies. We will require

$$\delta\langle\psi|\hat{H}|\psi\rangle = \langle\delta\psi|\hat{H}|\psi\rangle = 0 \tag{7.96}$$

to obtain the best ψ to minimize $\langle\hat{H}\rangle$ ($\hat{H}$ is the *full* Hamiltonian).

The most general variation we consider is a sum of 1-particle 1-hole states. Recall that the $\{\phi_\beta\}$ are a complete basis and divide the single particle states into two sets:

$$\begin{array}{l} \alpha,\ \beta, \ldots = \text{occupied states} \\ a, b, \ldots = \text{unoccupied states.} \end{array} \tag{7.97}$$

Our variation on the wave function ψ is then

$$\delta\psi = \sum_{a,\ \beta} \eta_{a\beta}\psi(a,\ \beta^{-1}) \tag{7.98}$$

and our variational minimization is

$$\sum_{\substack{a, b \ldots \\ \alpha, \beta, \ldots}} \eta_{a\beta}\langle\psi(a,\ \beta^{-1})|\hat{H}|\psi\rangle = 0. \tag{7.99}$$

This will yield the Hartree-Fock condition:

$$\langle a|\hat{T}|\beta\rangle + \sum_{\alpha\neq\beta}[\langle a\alpha|V|\beta\alpha\rangle - \langle a\alpha|V|\alpha\beta\rangle] = 0 \tag{7.100}$$

where one should note that the factor of $\frac{1}{2}$ has disappeared since we keep the exchange term (2nd potential term) explicitly.

As a simple example, it is useful to consider $A = 2$:

$$\delta\psi = \eta_{a\beta}\cdot\frac{1}{\sqrt{2}}\begin{vmatrix}\phi_\alpha(1) & \phi_\alpha(2)\\ \phi_a(1) & \phi_a(2)\end{vmatrix} + \eta_{a\alpha}\cdot\frac{1}{\sqrt{2}}\begin{vmatrix}\phi_a(1) & \phi_a(2)\\ \phi_\beta(1) & \phi_\beta(2)\end{vmatrix} \tag{7.101}$$

Since we can choose $\eta_{a\beta}$ and $\eta_{a\alpha}$ independently each term must vanish separately. Then we have

$$\frac{1}{2}\langle\phi_a(1)\phi_\beta(2) - \phi_\beta(1)\phi_a(2)|\hat{H}|\phi_\alpha(1)\phi_\beta(2) - \phi_\beta(1)\phi_\alpha(2)\rangle = 0 \tag{7.102}$$

$$\text{or} \qquad \frac{1}{2}\langle a\beta - \beta a|\sum_{i=1}^{2} T_i + \frac{1}{2}\sum_{i\neq j} V_{ij}|\alpha\beta - \beta\alpha\rangle = 0 \tag{7.103}$$

Then we use

$$\langle a\beta - \beta a|T_1|\alpha\beta - \beta\alpha\rangle = \langle a\beta - \beta a|T_2|\alpha\beta - \beta\alpha\rangle = \langle a|T|\alpha\rangle \tag{7.104}$$

$$\langle a\beta - \beta a|\frac{1}{2}V_{12}|\alpha\beta - \beta\alpha\rangle = \langle a\beta - \beta a|\frac{1}{2}V_{21}|\alpha\beta - \beta\alpha\rangle \tag{7.105}$$

$$= \frac{1}{2}[\langle a\beta|V_{12}|\alpha\beta\rangle + \langle\beta a|V_{12}|\beta\alpha\rangle - \langle\beta a|V_{12}|\alpha\beta\rangle - \langle a\beta|V_{12}|\beta\alpha\rangle] \tag{7.106}$$

$$= [\langle a\beta|V|\alpha\beta\rangle - \langle a\beta|V|\beta\alpha\rangle] \tag{7.107}$$

to obtain

$$\langle a|T|\alpha\rangle + [\langle a\beta|V|\alpha\beta\rangle - \langle a\beta|V|\beta\alpha\rangle] = 0. \tag{7.108}$$

For the general case $A > 2$ we obtain the Hartree-Fock (HF) condition

$$\langle a|T|\alpha\rangle + \sum_{\beta\neq\alpha}[\langle a\beta|V|\alpha\beta\rangle - \langle a\beta|V|\beta\alpha\rangle] = 0. \tag{7.109}$$

We can satisy the HF condition by solving the HF equations:

$$T|\phi_\alpha\rangle + \sum_{\beta\neq\alpha}[\langle\phi_\beta|V|\phi_\alpha\phi_\beta\rangle - \langle\phi_\beta|V|\phi_\beta\phi_\alpha\rangle] = E_\alpha|\phi_\alpha\rangle. \tag{7.110}$$

Therefore, if we choose this as our single particle problem $h|\phi_\alpha\rangle = E_\alpha|\phi_\alpha\rangle$, then the $|\phi_\alpha\rangle$ will satisfy the HF condition and represent the best single particle approx. solution to $\hat{H}\psi = E\psi$, where H is the full Hamiltonian. To see more clearly how to solve this problem, let's rewrite the HF equations in coordinate space

$$-\frac{\hbar^2}{2M}\nabla^2\phi_\alpha(r) + \int U(r',\ r)\phi_\alpha(r')d^3r' = E_\alpha\phi_\alpha(r) \tag{7.111}$$

where

$$U(r',\ r) = \sum_\beta\{\delta^3(r-r')\int\phi^*_\beta(r'')V(r'',\ r)\phi_\beta(r'')d^3(r'') - \phi^*_\beta(r')V(r',\ r)\phi_\beta(r)\} \tag{7.112}$$

is *best* effective single particle potential.

Note that knowing the best U requires we that we already know $\{\phi_\alpha\}$, the solution! So we need to solve the HF equations self-consistently:

- • Assume (guess) $\hat{h} \to \{\phi_\alpha\}\{E_\alpha\}$
- • Pick lowest A E_α, use $\{\phi_\alpha\}$ to obtain U
- • Now we have a new $\hat{H}_0 = \sum h_i$. Solve again for $\{\phi_\alpha\}$, $\{E_\alpha\}$ and iterate until the solution converges.

Then the final answer for the ground state energy is

$$E_{HF} = \langle\psi|\hat{H}|\psi\rangle = \sum_{\alpha=1}^{A}\langle\alpha|T|\alpha\rangle + \frac{1}{2}\sum_{\alpha\beta}^{A}[\langle\alpha\beta|V|\alpha\beta\rangle - \langle\alpha\beta|V|\beta\alpha\rangle] \tag{7.113}$$

$$= \sum_\alpha E_\alpha - \frac{1}{2}\sum_{\alpha\beta}[\langle\alpha\beta|V|\alpha\beta\rangle - \langle\alpha\beta|V|\beta\alpha\rangle]. \tag{7.114}$$

Nuclear Matter in Mean Field Approximation

We now consider *infinite* nuclear matter in the independent particle approximation. The most general single-particle Hamiltonian is

$$-\frac{\hbar^2}{2M}\vec{\nabla}^2\phi_\alpha(\vec{r}) + \int d^3r'U(\vec{r}'-\vec{r})\phi_\alpha(\vec{r}') = E_\alpha\phi_\alpha(\vec{r}) \tag{7.115}$$

where U may in principle depend on spin and isospin. The solutions are plane waves

$$\phi_\alpha = \frac{1}{\Omega}\psi_\alpha(S,\ T)e^{i\vec{k}_\alpha\cdot\vec{r}} \tag{7.116}$$

with a dispersion relation

$$\frac{-\hbar^2 k_\alpha^2}{2M} + U(k_\alpha) \qquad = \qquad E_\alpha \tag{7.117}$$

$$\left[U(k_\alpha) \quad \equiv \&intd^3\vec{r}U(r)\, e^{i\vec{k}_\alpha\cdot\vec{r}}\right] \tag{7.118}$$

The A-body ground state wave function is the Slater determinant

$$\psi(1,\ 2,\ldots A) = \frac{1}{\sqrt{A!}}\begin{vmatrix} \phi_\alpha(1) & \phi_\alpha(2) & \ldots & \phi_\alpha(A) \\ \phi_\beta(1) & \phi_\beta(2) & \ldots & \phi_\beta(A) \\ \vdots & \vdots & & \vdots \end{vmatrix} \tag{7.119}$$

where $\{\alpha,\ \beta\ldots\}$ are the lowest A single particle states. Even though it is a product of independent particles, $\psi(1,\ 2,\ldots A)$ has important Pauli correlations. Consider the probability density to find particle 1 at $\vec{r}_1$ and particle 2 at $\vec{r}_2$:

$$P_{\_}(\vec{r}_1,\vec{r}_2) \equiv \int \psi^*(1,\ldots A)\psi(1,\ldots A)\ d^3r_3 d^3r_4 \ldots d^3r_A. \tag{7.120}$$

$$= \frac{1}{A(A-1)}\cdot\frac{1}{2}\sum_{\alpha,\ \beta}|\phi_\alpha(1)\phi_\beta(2) - \phi_\beta(1)\phi_\alpha(2)|^2. \tag{7.121}$$

We temporarily ignore spin, isospin, so $\phi_\alpha = \frac{1}{\Omega}e^{i\vec{k}_\alpha\cdot\vec{r}}$. Then

$$P_{\_}(\vec{r}_1,\vec{r}_2) = \frac{1}{A(A-1)}\cdot\frac{\Omega^2}{2\Omega^2}\int\frac{d^3k_\alpha}{(2\pi)^3}\frac{d^3k_\beta}{(2\pi)^3}\left[e^{-i\vec{k}_\alpha\cdot\vec{r}_1}e^{-i\vec{k}_\beta\cdot\vec{r}_2} - e^{-i\vec{k}_\beta\cdot\vec{r}_1}e^{-i\vec{k}_\alpha\cdot\vec{r}_2}\right] \tag{7.122}$$

$$\times\left[e^{i\vec{k}_\alpha\cdot\vec{r}_1}e^{i\vec{k}_\beta\cdot\vec{r}_2} - e^{i\vec{k}_\beta\cdot\vec{r}_1}e^{i\vec{k}_\alpha\cdot\vec{r}_2}\right] \tag{7.123}$$

$$= \frac{1}{A(A-1)}\cdot\frac{1}{2}\int\frac{d^3k_\alpha}{(2\pi)^3}\frac{d^3k_\beta}{(2\pi)^3}\left[2 - e^{i(\vec{k}_\alpha-\vec{k}_\beta)\cdot(\vec{r}_1-\vec{r}_2)} - e^{-i(\vec{k}_\alpha-\vec{k}_\beta)\cdot(\vec{r}_1-\vec{r}_2)}\right] \tag{7.124}$$

$$= \frac{1}{A(A-1)}\int\frac{d^3k_\alpha}{(2\pi)^3}\frac{d^3k_\beta}{(2\pi)^3}\left[1 - \cos(\vec{k}_\alpha - \vec{k}_\beta)\cdot(\vec{r}_1 - \vec{r}_2)\right] \tag{7.125}$$

We now let $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$ and use

$$\cos[(\vec{k}_\alpha - \vec{k}_\beta)\cdot\vec{r}] = \cos(\vec{k}_\alpha\cdot\vec{r})\cos(\vec{k}_\beta\cdot\vec{r}) + \sin(\vec{k}_\alpha\cdot\vec{r})\sin(\vec{k}_\beta\cdot\vec{r}) \tag{7.126}$$

Now the sine terms vanish

$$\int d^3k\sin(\vec{k}\cdot\vec{r}) = 0 \tag{7.127}$$

so

$$P_{\_}(\vec{r}) = \frac{1}{A(A-1)}\times\frac{1}{(2\pi)^6}\left\{\left(\frac{4\pi k_f^3}{3}\right)^2 - \left[\int_0^{k_F} d^3k\cos(\vec{k}\cdot\vec{r})\right]^2\right\}. \tag{7.128}$$

We can evaluate the cosine integral as follows.

$$\int d^3\vec{k}\cos(\vec{k}\cdot\vec{r}) = \frac{2\pi}{2}\int k^2 dk\ d(-\cos\theta)\Big[e^{ikr\cos\theta}+e^{-ikr\cos\theta}\Big] \tag{7.129}$$

$$= \pi\int k^2 dk\Big[\frac{e^{ikr}-e^{-ikr}}{ikr}+\frac{e^{-ikr}-e^{ikr}}{-ikr}\Big] \tag{7.130}$$

$$= \frac{4\pi}{r}\int_0^{k_F} k\sin(kr)\ dk \tag{7.131}$$

$$= \frac{4\pi}{r}\left(-\frac{d}{dr}\right)\int_0^{k_F}\cos kr\ dk \tag{7.132}$$

$$= -\frac{4\pi}{r}\cdot\frac{d}{dr}\Big[\frac{\sin k_F r}{r}\Big] \tag{7.133}$$

$$= -\frac{4\pi}{r}\Big[-\frac{\sin k_F r}{r^2}+\frac{k_F\cos k_F r}{r}\Big] \tag{7.134}$$

$$= \frac{4\pi}{r^3}\Big[\sin(k_F r)-(k_F r)\cos(k_F r)\Big] \tag{7.135}$$

So we have the result

$$P_{\_}(r) = \frac{1}{A(A-1)}\frac{A^2}{\Omega^2}\Big\{1-\Big[\frac{3}{(k_F r)^2}\left(\frac{\sin k_F r}{k_F r}-\cos k_F r\right)\Big]^2\Big\} \tag{7.136}$$

$$= \frac{1}{\Omega^2}\Big[1-F_1(k_F r)\Big]. \tag{7.137}$$

"Classically", we would expect $P = P_1 P_2 = \frac{1}{\Omega}\cdot\frac{1}{\Omega}$, so F_1 is a correction for Pauli principle correlations.

Now consider the effects of spin-isospin. If one follows through the same analysis for spatially symmetric (rather than antisymmetric) case, one find

$$P_+(r) = \frac{1}{\Omega^2}[1+F_1(k_F r)]; \tag{7.138}$$

which applies for spin-isospin antisymmetric states. For a given state that is spatially symmetric (and antisymmetric in spin-isospin) we require $S=1,T=0$ or $S=0,T=1$ which gives a total of 6 spin-isospin states. For a spatially antisymmetric state, we require $S=0,T=0$ or $S=1,T=1$ and there are 10 such states. Therefore, we can write

$$P(r) = \frac{10}{16}P_-(r)+\frac{6}{16}P_+(r) \tag{7.139}$$

$$= \frac{1}{\Omega^2}\Big[1-\frac{1}{4}F_1(k_F r)\Big] \tag{7.140}$$

The function $F_1(x)$ is shown in Figure 7.7. Note that there is a "wound" in the correlation function for $r < \frac{2}{k_F} \approx 1.4$ fm.

7.6 Pair Correlation in Nuclear Matter

The independent pair approximation is a correction to the independent particle treatment discussed above. This approximation allows for 2-particle correlations but neglects correlations among clusters of more than two particles.

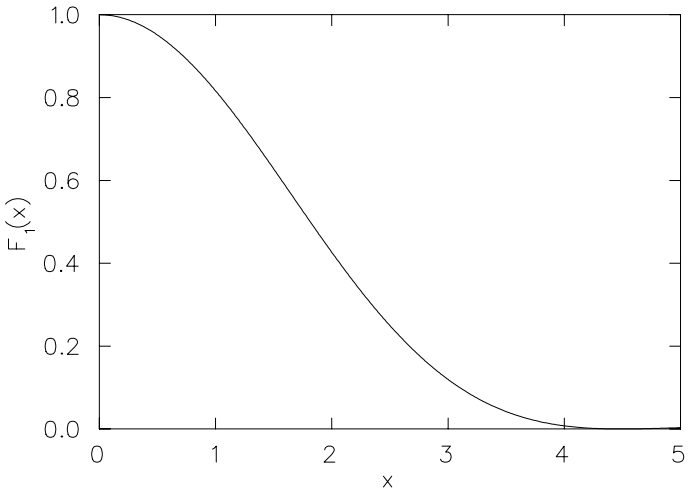

Figure 7.7: The correlation function F_1 vs. $x = k_F r$ showing the wound at small x.

We want to know the effect of the real 2 nucleon interaction V_{12} on the true 2 particle wave function ψ_{12}. Let ϕ_{12} be a Slater determinant, and let's define an operator $\hat{G}$:

$$\hat{G}\phi_{12} \equiv V_{12}\psi_{12}. \tag{7.141}$$

Thus $\hat{G}$ is an effective potential that operates on single particle states to result in the same state obtained by operating V_{12} in the true state ψ_{12}.

- • As before, we have the single particle Schrödinger Equation (SE)

$$h\phi_\alpha = (T + U_0)\phi_\alpha = E_\alpha\phi_\alpha \tag{7.142}$$

  which generates the single particle states ϕ_α.

- • The two particle SE is

$$[\hat{H}_0 + V_{12}]\psi_{kl}(1,2) = E_{kl}\psi_{kl}(1,2) \tag{7.143}$$

  where $\hat{H}_0 = h_1 + h_2$ and k, l correspond to $\phi_k\phi_l$ which are best approximation to ψ_{kl}.

- • Now expand ψ_{kl} in eigenfunctions of H_0:

$$\psi_{kl} = \phi_k(1)\phi_l(2) + \sum_{a,b} a_{ab}\phi_a(1)\phi_b(2) + \sum_a a_{ka}\phi_k(1)\phi_a(2) + \sum_a a_{al}\phi_a(1)\phi_l(2) \tag{7.144}$$

  where $\left.\begin{matrix} a, b, c \ldots \text{unoccupied} \\ k, l, m \ldots \text{occupied} \end{matrix}\right\}$ are single particle solutions to H_0. We rewrite this in a more compact notation as

$$|\psi_{kl}\rangle = |kl\rangle + \sum_{ab} a_{ab}|ab\rangle + \sum_a a_{ka}|ka\rangle + \sum_a a_{al}|al\rangle. \tag{7.145}$$

Now substitute in Equation 7.143 above and note that momentum conservation implies that

$$\langle \alpha\beta | V_{12} | kl \rangle = 0 \text{ for } \alpha = k \text{ or } \beta = l \text{ (not both).} \tag{7.146}$$

Thus we find that, for infinite nuclear matter, 1-particle 1-hole states do not contribute and we have only 2-particle 2-hole states. Then contract with $\langle ab|$ to obtain

$$(E_a + E_b)a_{ab} + \langle ab|V_{12}|\psi_{kl}\rangle = a_{ab}E_{kl} \tag{7.147}$$

$$a_{ab} = \frac{\langle ab|V_{12}|\psi_{kl}\rangle}{E_{kl} - E_a - E_b} \tag{7.148}$$

$$|\psi_{kl}\rangle = |kl\rangle + \sum_{ab} \frac{|ab\rangle\langle ab|V_{12}|\psi_{kl}\rangle}{(E_{kl} - \hat{H}_0)} \tag{7.149}$$

The amplitudes a_{ab} represent the population of 2-particle 2-hole states with both particles outside the degenerate ground state. That is, both particles have $E > E_F$ and $k > k_F$. The distribution of occupied states is thus modified as shown in Figure 7.8.

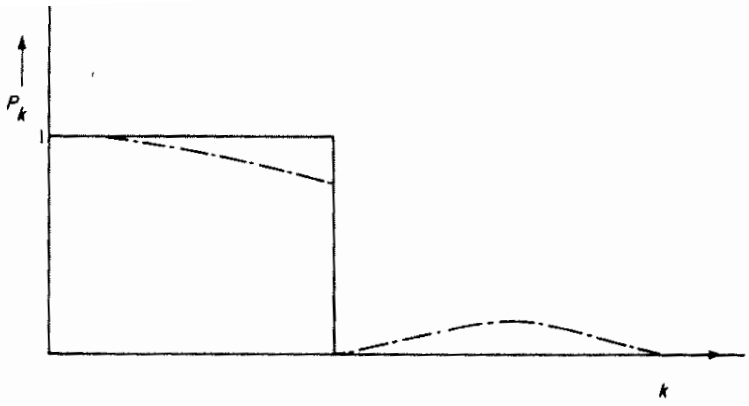

Fig. 8-8 A schematic representation of the state occupation number. The solid line is a sharp Fermi surface, $P_k = 1$, $k < k_F$, and zero otherwise. The dot-dash line shows the effect of correlations in particle occupancy. The shape of this curve depends on the nature of the two-body interaction. For $k > k_F$, P_k will be largest when k is near the maximum of the Fourier transform of $v(r)\zeta(r)$, as can be deduced from Eq. 8–41.

Figure 7.8: Schematic occupation probability in momentum space for the Fermi Gas (solid) and independent pair approximation (dashed).

Finally, we define the projection operator for initially unoccupied states

$$\hat{Q} \equiv \sum_{ab} |ab\rangle\langle ab| \tag{7.150}$$

and obtain the Bethe-Goldstone equation

$$|\psi_{kl}\rangle = |kl\rangle + \frac{\hat{Q}}{E_{kl} - \hat{H}_0} V_{12}|\psi_{kl}\rangle \tag{7.151}$$

The effective interaction then becomes

$$\hat{G} = V + V\frac{Q}{E_{kl} - \hat{H}_0}\hat{G}. \tag{7.152}$$

which has the important properties

- • depends on V
- • depends on $k_f(E_f)$ (through $\hat{Q}$)
- • depends on H_0 (U_0).

We now compute the energy

$$E = \sum_k \langle k|\hat{H}_0|k\rangle + \frac{1}{2}\sum_{kl}[\langle kl|G|kl\rangle - \langle kl|G|lk\rangle]. \tag{7.153}$$

as a function of k_F (density). The k_F that minimizes this energy is the ground state, and the corresponding density should be the density of nuclear matter in its ground state. The results of a typical calculation with two different N-N potential models is shown in Fig. 7.10.

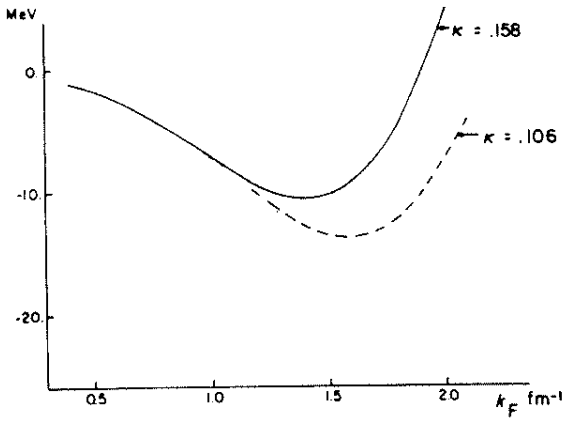

Fig. 8-12 Binding energy of nuclear matter as a function of Fermi momentum. Upper curve: Reid soft core, lower curve: OBEP of Grangé and Preston [17]. Two-body correlations only. The quantity κ is the wound at $k_F = 1.5$ fm^{-1}, $k = 0.86$ fm^{-1}.

Figure 7.9: Binding energy of nuclear matter as a function of Fermi momentum, for 2 values of the wound integral κ.

The effective interaction (G) between nucleons in the nuclear medium is generally very much weaker than the N-N interaction in free space. This is illustrated in the figure below.

As a result, it is then a very good approximation to consider the nucleons as essentially free particles within the nuclear volume.

7.7 Problems

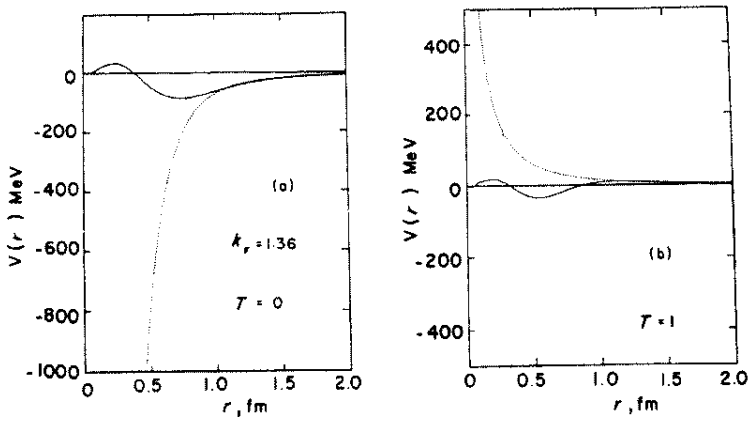

Fig. 8-14 Tensor components of the effective interaction at $k_F = 1.36$ fm^{-1} for $T = 0$ and 1. For comparison, we also show, as a dotted curve, corresponding components of the free potential, from which the effective interaction was derived. (From ref. 38.)

Figure 7.10: Effective tensor interactions for infinite nuclear matter.