Chapter 6

Nucleon-Nucleon Interaction, Deuteron

Protons and neutrons are the lowest-energy bound states of quarks and gluons. When we put two or more of these particles together, they interact, scatter and sometimes form bound states due to the strong interactions. If one is interested in the low-energy region where the nucleons hardly get excited internally, we can treat the nucleons as inert, structureless elementary particles, and we can understand many of the properties of the multi-nucleon systems by the nucleon-nucleon interactions. If the nucleons are non-relativistic, the interaction can be described by a potential. Since the fundamental theory governing the nucleon-nucleon interactions is QCD, the interactions shall be calculable from the physics of quarks and gluons. Nonetheless, the problem is quite complicated and only limited progress has been made from the first principles so far. Therefore, the approach we are going take is a phenomenological one: One first tries to extract the nucleon-nucleon interaction from the nucleon-nucleon scattering data or few nucleon properties, and then one tries to use these interactions to make predictions for the nuclear many-body system. In this and the following few chapters, we are going to discuss some of results of this approach.

6.1   Yukawa Interaction and One-Pion Exchange at Long Distance

If we consider the nucleons as elementary particles, their strong interactions are known have a short range from α particle scattering experiments of Rutherford. In fact, the range of the interaction is roughly the size of the atomic nuclei, namely on the order of a few fermis (fm) (1 fm = 10^{-15}m).

The first theory of the nucleon force was put forward by H. Yukawa, who suggested that the interaction between two nucleons is effected by the exchange of a particle, just like the interaction between the electric charges by the exchange of a photon. However, because the nucleon interactions appear to be short-ranged, the particle must have a finite mass. In fact, one can correlate the range and mass roughly by the quantum uncertainty principle

$$r \sim 1/m \; , \tag{6.1}$$

therefore, the mass of the quanta exchanged is about 1/fm which is about 200 MeV. The particle was discovered almost 20 years later and was identified as π meson (140 MeV). The most significant aspect of the Yukawa theory is generalizing the relation between particles and forces—-the existence

of strong interactions implies the existence of a new particle! This was considered a novel and radical idea at that time.

The modern theory of interactions through particle exchanges is made possible by the development of quantum field theory. However, at low-energy, one can assume the interactions is instantaneous and therefore the concept of interaction potential becomes useful. To illustrate the derivation of a potential through particle exchange, consider the interaction of a scalar particle with the nucleons. Introducing a quantum field ϕ to represent the particle, the Yukawa interaction has the form,

$$\mathcal{L} = g\phi\bar{\psi}\psi \ . \tag{6.2}$$

The scattering S-matrix between two nucleon states with momentum p_1 and p_2 is

$$S = (2\pi)^4\delta^4(p_1 + p_2 - p_1' - p_2')ig\overline{U}(p_1')U(p_1)\frac{i}{(p_1 - p_1')^2 - m^2 + i\epsilon}ig\overline{U}(p_2')U(p_2) \tag{6.3}$$

An interacting potential can be derived with the following approximation: Since particles are non-relativistic, their energy is dominated by the rest mass, and therefore

$$(p_1 - p_1')^2 = -(\vec{p} - \vec{p}')^2 + \mathcal{O}(1/M) \ , \tag{6.4}$$

where M is the nucleon mass. The propagator is now energy-independent, and the interaction is instantaneous and can be represented by a potential. The on-shell Dirac spinor is $U = \sqrt{E_p + M}(\chi, \vec{\sigma}\cdot\vec{p}/(E_p + M)\chi)$ which to the leading order can be approximated by $U = \sqrt{2M}(\chi, 0)$. Therefore the S-matrix can be approximated by

$$S \sim -i(2\pi)^4\delta^4(p_1 + p_2 - p_1' - p_2')(2M)^2\chi_{2'}^\dagger\chi_2\chi_{1'}^\dagger\chi_1\frac{-g^2}{\vec{q}^2 + m^2} \tag{6.5}$$

where we introduce the center-of-mass and relative variables,

$$\begin{aligned}
\vec{R} &= (\vec{r}_1 + \vec{r}_2)/2, \quad \vec{r} = \vec{r}_1 - \vec{r}_2; \\
\vec{P} &= \vec{p}_1 + \vec{p}_2, \quad \vec{p} = (\vec{p}_1 - \vec{p}_2)/2;
\end{aligned} \tag{6.6}$$

and assume $\vec{P} = 0$. Therefore, $\vec{q} = \vec{p} - \vec{p}'$ is the relative momentum.

The relationship between the interaction potential V and the S-matrix is

$$S = \exp(-i\int dtV) \sim -i\int dtV = -i(2\pi)\delta(0)V \tag{6.7}$$

Therefore, we find

$$\langle\vec{p}_1'\vec{p}_2'|V|\vec{p}_1\vec{p}_2\rangle = (2\pi)^3\delta(\vec{p}_2 + \vec{p}_1 - \vec{p}_2 - \vec{p}_1)\chi_{2'}^\dagger\chi_2\chi_{1'}^\dagger\chi_1\frac{-g^2}{\vec{q}^2 + m^2} \tag{6.8}$$

where the states are normalized according to $\langle\vec{p}|\vec{p}\rangle = \chi^\dagger\chi(2\pi)^3\delta(\vec{p} - \vec{p}')$. The δ-function in the above matrix element indicates that the interaction is translational-invariant, and V is a function of the relative coordinate $\vec{r}$. The above matrix element is then

$$\int e^{-\vec{q}\cdot\vec{r}}V(r)d^3\vec{r} = \frac{-g^2}{\vec{q}^2 + m^2} \ . \tag{6.9}$$

Making an inverse Fourier transformation, one gets

$$V(r) = -\frac{g^2}{4\pi}\frac{e^{-mr}}{r} \; , \tag{6.10}$$

which is an attractive interaction, independent of the spin of the nucleon. When $m = 0$, the interaction is Coulomb-like and is long range. For a finite m, the interaction is negligible beyond the distance $r \sim 1/m$.

Pi-meson is the lightest meson because of the spontaneous chiral symmetry breaking. It is responsible to the longest range nucleon-nucleon force. One of the important new features here is isospin symmetry. Pi-meson has iso-spin 1, and can be represented by a vector $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ in isospin space. Combinations of $\pi_{1,2}$ generate positive and negative charged pions $\pi^\pm$, and π_3 corresponds to a neutral pion π^0. Using ψ to represent the nucleon field, a spinor in the isospin space $\psi = (p, n)$, a isospin symmetric Yukawa interaction is

$$\mathcal{L}_Y = g_{\pi NN}\overline{\psi}i\gamma_5\vec{\tau}\psi \cdot \vec{\phi} \tag{6.11}$$

where $\vec{\pi}$ is the isovector pion field.

We can again consider the nucleon-nucleon interaction through one-pion exchange. Because the pion is a pseudo-scalar particle, parity conservation requires that it be emitted or absorbed in the p-wave. Therefore there is a factor of $\vec{p}$ associated with the interaction vertex in the momentum space. This factor is multiplied by the spin operator $\vec{sigma}$ forms a pseudo-scalar. Going through the same derivation as the scalar case, we find the following potential,

$$V(\vec{r}) = \frac{g^2}{4M_N^2}(\vec{\tau}_1 \cdot \vec{\tau}_2)(\vec{\sigma}_1 \cdot \vec{\nabla})(\vec{\sigma}_2 \cdot \vec{\nabla})\frac{e^{-m_\pi r}}{r} \; . \tag{6.12}$$

where the isospin factor $\vec{\tau}_1 \cdot \vec{\tau}_2$ depends on the isospin states of the two nucleons.

For a system of two interacting nucleons the total isospin operator is given by

$$\vec{T} = \vec{t}_1 + \vec{t}_2 \; . \tag{6.13}$$

If we ignore the EM interaction and the mass difference of up and down quarks, the interaction Hamiltonian conserves isospin and so commutes with all components of isospin

$$[\hat{H}, \vec{T}] = 0 \; . \tag{6.14}$$

Then since $\hat{H}$ is invariant under rotations in isospin space, it can only depend on the isospin through $\vec{T}^2$, where

$$\begin{aligned}
\vec{T}^2 &= (\vec{t}_1 + \vec{t}_2)^2 & (6.15)\\
&= \vec{t}_1^2 + \vec{t}_2^2 + 2\vec{t}_1 \cdot \vec{t}_2 & (6.16)\\
&= \frac{3}{4} + \frac{3}{4} + \frac{1}{2}\vec{\tau}_1 \cdot \vec{\tau}_2 & (6.17)
\end{aligned}$$

So $\hat{H}$ or potential can be a function of the quantity $\vec{\tau}_1 \cdot \vec{\tau}_2$.

The states of good total $\hat{T}^2$ are also eigenstates of $\hat{H}$. Then we expect the following properties of a system of 2 nucleons:

- The eigenstates with $T = 1$ will be degenerate (just as the $S = 1$ states for a system of 2 spin $\frac{1}{2}$ objects with a rotationally symmetric Hamiltonian), i.e., the states

$$pp \qquad nn \qquad \frac{1}{\sqrt{2}}(pn + np) \tag{6.18}$$

  all have identical interactions and will be degenerate in energy.

- The $T = 0$ combination $\frac{1}{\sqrt{2}}(pn - np)$ can be different (and generally is), i.e, its energy is different from the 3 $T = 1$ states and the interaction differs in this channel.

It is therefore advantageous to form states of good isospin to describe a system of 2 nucleons. Note the combination $\vec{\tau}_1 \cdot \vec{\tau}_2$ is given by

$$\vec{\tau}_1 \cdot \vec{\tau}_2 = 4\vec{T}_1 \cdot \vec{T}_2 = 2\left[T(T+1) - \frac{3}{2}\right] = \begin{cases} -3, \, T = 0 \\ 1, \, T = 1 \end{cases}. \tag{6.19}$$

Going back to the one-pion exchange potential, it can be shown that

$$(\vec{\sigma}_1 \cdot \vec{\nabla})(\vec{\sigma}_2 \cdot \vec{\nabla})\frac{e^{-m_\pi r}}{m_\pi r} = \frac{1}{3}m_\pi^2 \left[\vec{\sigma}_1 \cdot \vec{\sigma}_2 + \hat{S}_{12}\left(1 + \frac{3}{m_\pi r} + \frac{3}{m_\pi^2 r^2}\right)\right]\frac{e^{-m_\pi r}}{m_\pi r}. \tag{6.20}$$

where S_{12} is called tensor operator, defined as

$$S_{12} = 2\left[3\frac{(\vec{S} \cdot \vec{r})^2}{r^2} - \vec{S}^2\right] \tag{6.21}$$

where $\vec{S}$ is the total spin.

Putting together all the factors, one finds that the nucleon-nucleon potential from one pion exchange as,

$$V = V_0(\vec{\tau}_1 \cdot \vec{\tau}_2)\left[\vec{\sigma}_1 \cdot \vec{\sigma}_2 + S_{12}\left(1 + \vec{3}m_\pi r + \vec{3}(m_\pi r)^2\right)\right]\frac{e^{-m_\pi r}}{m_\pi r} \tag{6.22}$$

where $V_0 = g^2 m_\pi^2/12M^2$. This potential matches the phenomenological forms extracted from experimental data at large N-N separations ($\sim 2 - 3$ fm).

At smaller distance, there is also exchanges from scalar meson (isospin 0) of about 500 MeV. The interaction is attractive as we seen above, corresponding to a medium range attraction. Finally there are also exchanges from vector mesons, ω (isospin-0) meson and ρ meson (spin-1). The interactions from the ρ and ω meson are short-range repulsive. One can build a phenomenological nucleon-nucleon interaction based on the meson exchanges. The so-called Nijmegen potential and Bonn potential are generated through this approach. The short range interactions are model dependent by nature, there is no unique picture for them. This is so because only in the low-energy processes, potentials are useful concept, however, the short range interactions are not so sensitive to the low-energy observables. Therefore, either one can adopt a more phenomenological approach in which some "educated" functional forms are assumed or use the so-called effective field theory approach to parameterize the unknowns systematically.

6.2 General Aspects of the Two-Body Interactions

For the two-nucleon system, the experimental information consists of two particle scattering phase shifts in various partial waves, and the bound state properties of the deuteron. From data, one learns that the nuclear force has the following the main features:

- Attractive: to form nuclear bound states

- Short Range: of order 1 fm

- Spin-Dependent:

- Noncentral: there is a tensor component)

- Isospin Symmetric:

- Hard Core: so that the nuclear matter does not collapse

- Spin-Orbit Force

- Parity Conservation

In phenomenology, one tries to come up with the forms of the forces which will satisfy the above properties. In particular, one parametrize the short distance potentials consistent with fundamental symmetries and fit the parameters to experimental data.

For the system of two nucleons, use $\vec{r} = \vec{r}_1 - \vec{r}_2$ to represent relative position and $\vec{p} = (\vec{p}_1 - \vec{p}_2)/2$ relative momentum, $\vec{s}_1$ and $\vec{s}_2$ their respective spins. The relative orbital angular momentum is $\vec{L} = \vec{r} \times \vec{p}$ and the total spin is $\vec{S} = \vec{s}_1 + \vec{s}_2$. When the spins are coupled, the total spin can either be 0 or 1. For the case of $S = 0$, we have a single spin state which is called *singlet*. For the case of $S = 1$, we have three spin states which are called *triplet*. The total angular momentum is the sum of orbital angular momentum and total spin: $\vec{J} = \vec{L} + \vec{S}$, which involves the coupling of three angular momenta $\vec{s}_1$, $\vec{s}_2$ and $\vec{L}$.

The orbital angular momentum quantum number is L. In the singlet spin case, we have $J = L$ because $S = 0$. For the triplet states, $J = L - 1, L, L + 1$ if $L \neq 0$, and $J = 1$ if $L = 0$. A state with (S, L, J) is usually labelled as $^{2S+1}L_J$, where $L = 0, 1, 2, 3$, are usually called $S, P, D, F, G...$ states. Therefore, we can make the following table for the angular momentum states,

$$
\begin{array}{ccccc}
J = 0 & & & ^3P_0 & ^1S_0 \\
J = 1 & ^3S_1 & ^3D_1 & ^3P_1 & ^1P_1 \\
J = 2 & ^3P_2 & ^3F_2 & ^3D_2 & ^1D_2 \\
J = 3 & ^3D_3 & ^3G_3 & ^3F_3 & ^1F_3
\end{array}
\tag{6.23}
$$

Because of SO(3) symmetry, no nuclear interactions can couple states with different total angular momenta. We say J is a good quantum number. Only the states in the first two columns can mix because they have the same parity and angular momentum. We choose the basis states with good J, $|n(LS)JM\rangle$ where,

$$
|(LS)JM\rangle = \sum_{m_L} Y_{Lm_L} |Sm_S\rangle \langle LmSm_S|JM\rangle
\tag{6.24}
$$

where $\langle LmSm_S|JM\rangle$ is the Clebsch-Gordon coefficient. In the coordinate representation, we use $\mathcal{Y}_{LSJ}^M$ to label the above total angular momentum eigenstates. Therefore, the eigenfunction can be written as

$$\psi_{nLSJM} = R_{n(SL)J}(r)\mathcal{Y}_{LSJ}^M .\tag{6.25}$$

where $R_{n(SL)J}(r)$ is the corresponding radial wave function.

Let us consider the constraint from Pauli principle. For the two nucleon system, the isospin T can either be 1 or 0. Since the total wave function has to be antisymmetric, let us consider what are the possible value of T, S, and L to make that happen. If we use $+1$ to represent symmetric wave function and -1 to represent antisymmetric, then the spin wave function is $(-1)^{S+1}$ and the isospin wave function has symmetry factor $(-1)^{T+1}$ and the orbital wave function is $(-1)^L$. The total symmetry factor is $(-1)^{L+S+T}$ which has to be -1. Therefore $L + T + S$ has to be odd. For deuteron, $S = 0, L = 0$ and therefore $T = 1$.

What are the possible forms of the nuclear force? The simplest is a central force which just dependent on the relative distance $V_C(r)$. In this case, different L states have different energies. For every L, the singlet and triplet spin states have the same energy. The eigen-functions of the system can chosen to be $|nLm_LSm_S\rangle \sim R_{nL}(r)Y_{LM_L}\chi_{SM_S}$.

There could be also a pure spin-dependence force. The most general form is $V_S(r)\vec{\sigma}_1 \cdot \vec{\sigma}_2$. In fact, we can write

$$\vec{\sigma}_1 \cdot \vec{\sigma}_2 = 2\vec{S}^2 - 3 .\tag{6.26}$$

Therefore the matrix element of the spin operator depends on the total spin of the two particles. In the singlet state, we have $\vec{\sigma}_1 \cdot \vec{\sigma}_2 = -3$, the potential is $V_C - 3V_S$; in the triplet state (S=1), we have $\vec{\sigma}_1 \cdot \vec{\sigma}_2 = 1$, and the potential is $V_C + V_S$. Now, the energy not only depends on the quantum number L but also on S. However, the eigen-function of the system can still be chosen as $|nLm_LSm_S\rangle$; the radial wave function depends on S, $R_{nLS}(r)$, because the potential does.

There can be also a pure iso-spin-dependence force. The most general form is $V_I(r)\vec{\tau}_1 \cdot \vec{\tau}_2$; There can be a spin-isospin dependent force. The most general form is $V_{SI}(r)\vec{\sigma}_1 \cdot \vec{\sigma}_2\vec{\tau}_1 \cdot \vec{\tau}_2$.

The spin-order force can be written as $V_{LS}(r)\vec{L} \cdot \vec{S}$. There can be higher-order spin-orbit terms like $(\vec{L} \cdot \vec{S})^n$ as well. From the angular momentum addition, we can write

$$\vec{L} \cdot \vec{S} = \frac{1}{2}(\vec{J}^2 - \vec{L}^2 - \vec{S}^2) .\tag{6.27}$$

Therefore the matrix element of the operator is simple in the basis which is formed by the common eigenstates of $\vec{J}^2, J_3, \vec{L}^2, \vec{S}^2$. The potential used for solving the radial Schroedinger equation is $V_{LSJ}(r) = V_C(r) + [2S(S + 1) - 3]V_S(r) + +[2T(T + 1) - 3]V_I(r) + [2S(S + 1) - 3][2T(T + 1) - 3]V_{SI}(r) + \frac{1}{2}[J(J + 1) - L(L + 1) - S(S + 1)]V_{LS}(r)$.

Finally, there is an interaction of the type

$$V_T(r)\left[3\frac{(\sigma_1 \cdot \vec{r})(\sigma_2 \cdot \vec{r})}{r^2} - \vec{\sigma}_1 \cdot \vec{\sigma}_2\right]\tag{6.28}$$

which is called the tensor interaction and the associated structure is often denoted as S_{12}. Using the total spin operator, we can write

$$(\vec{\sigma}_1 \cdot \vec{r})(\vec{\sigma}_2 \cdot \vec{r}) = 2(\vec{S} \cdot \vec{r})^2 - r^2\tag{6.29}$$

which can be proved by evaluating $(\vec{S} \cdot \vec{r})^2$. Then the tensor structure becomes $S_{12} = 2\left[3\frac{(\vec{S} \cdot \vec{r})^2}{r^2} - \vec{S}^2\right]$. It is not difficult to see that S_{12} is a scalar tensor operator arising from the coupling of the second rank tensors $(S^{(1)} \otimes S^{(1)})_m^{(2)}$ and Y_{2m}.

To determine how does S_{12} act on the angular momentum states, let us first show $Q = \frac{(\vec{S} \cdot \vec{r})^2}{r^2}$ is a projection operator. It is easy to show

$$
\begin{aligned}
Q &= \frac{1}{2}(1 + \vec{\sigma}_1 \cdot \hat{r}\vec{\sigma}_2 \cdot \hat{r}) \\
Q^2 &= Q
\end{aligned}
\tag{6.30}
$$

Then the tensor operator $S_{12} = 2[3Q - \vec{S}^2]$ satisfies the following equation,

$$
S_{12}^2 = 4\vec{S}^2 - 2S_{12} .
\tag{6.31}
$$

Therefore, its possible eigenvalues are 0, 2, −4. The action of $\hat{Q}$ on the total angular momentum eigenstates are then

$$
\begin{aligned}
Q \, \mathcal{Y}_{J0J}^M &= 0 \\
Q \, \mathcal{Y}_{J1J}^M &= \mathcal{Y}_{J1J}^M \\
(2J+1)Q \, \mathcal{Y}_{J+11J}^M &= J\mathcal{Y}_{J+11J}^M + \sqrt{J(J+1)}\mathcal{Y}_{J-11J}^M \\
(2J+1)Q \, \mathcal{Y}_{J-11J}^M &= \sqrt{J(J+1)}\mathcal{Y}_{J+11J}^M + (J+1)\mathcal{Y}_{J-11J}^M
\end{aligned}
\tag{6.32}
$$

There equations are nontrivial to show but are important for deriving the solution of the Schrodinger equation.

In the presence of the tensor interaction, $\vec{L}^2$ and $\vec{S}^2$ no longer commute with the hamiltonian. The states with the same J but different L can mix under the interaction. However, parity is still conserved, therefore, the states can mix only when their orbital angular momenta differ by 2 unit. The parity of an orbital angular momentum eigenstate is $(-)^L$, and is +1 when L is even and −1 when L is odd.

6.3 S-Wave Scattering

At low-energy (below 10 MeV or so), the dominant nucleon-nucleon scattering happens in the S-wave. In the zero-energy limit, the nucleon scattering cross section is huge, on the order of 20b or so. In this section, we consider the physics of the S-wave scattering.

In the region outside of the nuclear interaction, the S-wave scattering state is described by the solution of the free Schrodinger equation,

$$
\psi = A\sin(kr + \delta) \sim (e^{2i\delta}e^{ikr} - e^{-ikr})
\tag{6.33}
$$

where δ is the S-wave scattering phase-shift. Here the first exponential represents the outgoing wave and the first represents the incoming one. The total scattering cross section is

$$
\sigma = \frac{4\pi}{k^2}\sin^2\delta .
\tag{6.34}
$$

Since the cross section shall be finite as $k \to 0$ (the nuclear force has finite range), the phase shift shall vanish as $k \to 0$. In fact, at low-energy one can make the expansion according to the unitary condition,

$$k \cot \delta = -\frac{1}{a} + \frac{1}{2} r_0 k^2 + ... \tag{6.35}$$

where a is the scattering length and r_0 is the effective range, which roughly corresponds to the size range of the potential. At low-energy, the nuclear scattering can effectively determined by these two numbers. In fact the zero-energy cross section

$$\sigma = 4\pi a^2 . \tag{6.36}$$

which is completely determined by the scattering length.

The scattering lengths for nucleons have been extracted from scattering of neutron on proton and proton on proton. In the former case, there are both $T = 1, 0$ channels. The experimental result is

$$a_{S=0} = -23.7 \text{ fm} , \tag{6.37}$$

$$a_{S=1} = 5.38 \text{ fm} . \tag{6.38}$$

Therefore the experimental cross section is very large at zero-energy. In fact, the total neutron scattering cross section is $20b$! The result for $T = 1$, $S = 0$ channel obtained for the proton-proton scattering gives -17 fm, which is relatively close.

The large scattering length in fact indicates there is a two-body bound state, or quasi-bound state. Indeed for small but finite k, one has

$$f = -\frac{1}{1/a + ik}, \quad \sigma = \frac{4\pi}{1/a^2 + k^2} \tag{6.39}$$

The scattering matrix has a pole at $E = -\hbar^2/2a^2m^2$. If a is positive, this is a real bound state; if a is negative, the state is a virtual, sitting on the second sheet in the complex energy plane. Therefore, in the $S = 1$ channel, there must be a bound state, which is the deuteron. For $T = 1$, one has a virtual state: the potential is attractive but not attractive enough. A slight change of the potential might lead to a bound state.

6.4   Deuteron Structure

The deuteron is the only bound state of 2 nucleons, with isospin $T = 0$, spin-parity $J^\pi = 1^+$, and binding energy E_B=2.225 MeV. For two spin $\frac{1}{2}$ nucleons, only total spins $S = 0, 1$ are allowed. Then the orbital angular momentum is restricted to $J - 1 < l < J + 1$, i.e., $l = 0$, 1 or 2. Since the parity is $\pi = (-)^l = +$, only $l = 0$ and $l = 2$ are allowed; this also implies that we have $S = 1$.

If the hamiltonian is

$$H = -\frac{\hbar^2}{M} \frac{1}{r} \frac{d^2}{dr^2} r + \frac{\hbar^2}{M} \frac{L^2}{r^2} + V_C(r) + V_T(r) S_{12} \tag{6.40}$$

using the following relation,

$$S_{12} \mathcal{Y}_{001}^M = \sqrt{8} \mathcal{Y}_{211}^M; \quad S_{12} \mathcal{Y}_{211}^M = \sqrt{8} \mathcal{Y}_{011}^M - 2 \mathcal{Y}_{211}^M$$
$$L^2 \mathcal{Y}_{011}^M = 0; \qquad \vec{L}^2 \mathcal{Y}_{211}^M = 6 \mathcal{Y}_{211}^M \tag{6.41}$$

we find the radial equations

$$\left[\frac{\hbar^2}{M}\frac{d^2}{dr^2} + E - V_c(r)\right] u_S = \sqrt{8}V_T(r)u_D$$

$$\left[\frac{\hbar^2}{M}\left(\frac{d^2}{dr^2} - \frac{6}{r^2}\right) + E + 2V_T(r) - V_c(r)\right] u_D = \sqrt{8}V_T(r)u_S \tag{6.42}$$

These equations can be solved numerically.

Other important information on the structure of the deuteron comes from the values of the magnetic moment μ and quadrupole moment Q:

$$\mu = 0.8574 \ \mu_N \tag{6.43}$$

$$Q = 0.2857 \ e-\text{fm}^2 \tag{6.44}$$

Since Q$\neq$0, the deuteron cannot be pure $l = 0$. But generally $l = 0$ is energetically favored for a central potential. Therefore, we write the deuteron wave function as a linear combination of S- and D- waves

$$\psi(\vec{r}) = a\psi_{3S_1}(\vec{r}) + b\psi_{3D_1}(\vec{r}) \tag{6.45}$$

$$= [aR_0(r)\mathcal{Y}_{011}^1 + bR_2(r)\mathcal{Y}_{211}^1]\psi_{00}^T \tag{6.46}$$

where a and b are constants with $\sqrt{a^2 + b^2} = 1$. R_0 and R_2 are the radial wave functions, the isospin wave function is written as

$$\psi_{00}^T = \frac{1}{\sqrt{2}}[\chi_p(1)\chi_n(2) - \chi_n(1)\chi_p(2)] \tag{6.47}$$

6.4.1 Magnetic Moment

We first consider the implications of the magnetic moment. The magnetic moment operator is

$$\mu = \mu_N \sum_i (g_s s_{zi} + g_l l_{zi}) \tag{6.48}$$

where $g_s = 4.7\tau_i + 0.88$, where the first term is isovector, and the second term is isoscalar. $g_l = (\tau_i + 1)/2$. Since the deuteron is an iso-scalar particle, let us consider only the iso-scalar magnetic moment. Then, the above equation becomes, $\mu = \mu_N \sum_i(0.88s_{zi} + 0.5l_{zi})$.

Note that since $T = 0$ only the isoscalar magnetic moment operator contributes to μ:

$$\mu = \mu_N \sum_{i=1}^{2}\left[0.88\langle S_i^z\rangle_{M=1} + \frac{1}{2}\langle l_i^z\rangle_{M=1}\right] \tag{6.49}$$

$$= \mu_N\left[0.88\langle S_z\rangle + \frac{1}{2}\langle L_z\rangle\right]$$

$$= \mu_0\left[0.38\langle S_z\rangle + \frac{M}{2}\right] = \mu_N\left[0.38\langle S_z\rangle + \frac{1}{2}\right]. \tag{6.50}$$

where we have used the fact that the sum of the two orbital angular momenta can be decomposed into the sum of the center-of-mass angular momentum and relative angular momentum. The center-of-mass angular momentum give no contribution.

Let us now calculate the matrix element of S_z

$$
\begin{aligned}
\langle \mathcal{Y}_{011}^1 | S_Z | \mathcal{Y}_{011}^1 \rangle &= 1 \\
\langle \mathcal{Y}_{211}^1 | S_Z | \mathcal{Y}_{011}^1 \rangle &= 0 \\
\langle \mathcal{Y}_{211}^1 | S_Z | \mathcal{Y}_{211}^1 \rangle &= \sum_{M_S} |\langle 2(1 - M_S)1M_S|11\rangle|^2 \\
&= -1/2
\end{aligned}
\tag{6.51}
$$

Thus, for pure $l = 0$ or $l = 2$ states we would have the values $\mu = 0.88\mu_N, 0.31\mu_N$. More generally we obtain the relation

$$
\mu = [a^2(0.88) + b^2(0.31)]\mu_0 = (0.88 - 0.57b^2)\mu_0 .
\tag{6.52}
$$

Therefore, the experimental value $\mu_D = 0.857\mu_N$ implies that $b^2 = 0.04$. However, in more sophisticated treatments one finds that it is quantitatively important to explicitly include the effects of meson exchanges on the magnetic moment. For example, the virtual photon can couple to the pion in flight between the nucleons and convert it to a ρ meson. (These effects are clearly observed in elastic electron deuteron scattering at higher momentum transfers, as discussed below.) Including the effects of these "meson-exchange" currents on the deuteron's magnetic moment yields values of $b^2 = .05 - .07$.

6.4.2   Quadrupole Moment

Next we consider the quadrupole moment of the deuteron. Using the definition of $Q = \sqrt{\frac{16\pi}{5}}\langle J(M = J)|\hat{Q}_{20}|J(M = J)\rangle$, we can write

$$
Q = e\sqrt{\frac{16\pi}{5}} \int \psi_{J=M=1}^*(\vec{r})\left[\sum_{i=1}^{2}\frac{\tau_{3i} + 1}{2}r_i^2 Y_{20}(\hat{r}_i)\right]\psi_{J=M=1}(\vec{r})d^3\vec{r}
\tag{6.53}
$$

$$
= e\sqrt{\frac{16\pi}{5}} \int \psi_{J=M=1}^*(\vec{r})\frac{r^2}{4}Y_{20}(\hat{r})\psi_{J=M=1}(\vec{r})d^3r .
\tag{6.54}
$$

Here we have used the fact that for each nucleon the distance from the center of mass is only half the distance between them, $r_i = \frac{r}{2}$. Now using the expressions for the wave function introduced above

$$
Q = e\sqrt{\frac{16\pi}{5}}\left\{|a^2|\left(\int\frac{r^2}{4}R_0^2 r^2 dr\right)\int Y_{00}^* Y_{20}Y_{00}d\Omega + 2Re(ab^*)\left(\int\frac{r^2}{4}R_0 R_2 r^2 dr\right)\right.
\tag{6.55}
$$

$$
\times \sum_M \langle S = 1, M_S = 1|S = 1, M_S = 1 - M\rangle \int Y_{00}^* Y_{20}Y_{2M}d\Omega \times \langle 1(1 - M)2M|11\rangle
\tag{6.56}
$$

$$
\left. +|b|^2\left(\int\frac{r^2}{4}R_2^2 r^2 dr\right)\times\sum_M|\langle 1(1 - M)2M|11\rangle|^2\rangle\int Y_{2M}^* Y_{20}Y_{2M}d\Omega\right\} .
\tag{6.57}
$$

After evaluating the angular integrals and putting in the CG coefficients, one finds

$$
Q = e\left\{\frac{\sqrt{2}}{10}Re(ab^*)\left(\int r^4 R_0 R_2\right) - \frac{|b|^2}{20}\left(\int r^4 R_2^2\right)\right\} .
\tag{6.58}
$$

To proceed further we need to evaluate the radial integrals, so we would need to solve the radial Schrodinger equation and obtain the radial wave functions. Clearly, for a given potential model this is (in principle) possible. For our purposes, we will use our knowledge that $b = 0.2 \ll 1$ from the magnetic moment analysis and keep only the first term. This will give us an approximate expression that we can set equal to the experimental value $Q_{\text{exp}} = 0.286e$ fm^2 to obtain the result

$$Q \cong e\frac{(0.2) \cdot \sqrt{2}}{10} \int r^4 R_0 R_2 dr = 0.286 \ e \text{ fm}^2 \tag{6.59}$$

Solving for the unknown radial integral yields

$$\int r^4 R_0 R_2 dr \cong 10.1 \text{ fm}^2 \tag{6.60}$$

for the radial integral. This value seems quite reasonable given that the mean squared charge radius of the deuteron is 4.0 fm^2. [Note: $\langle r_{ch}^2 \rangle = \frac{1}{4} \int R_0^2 r^4 dr$.]
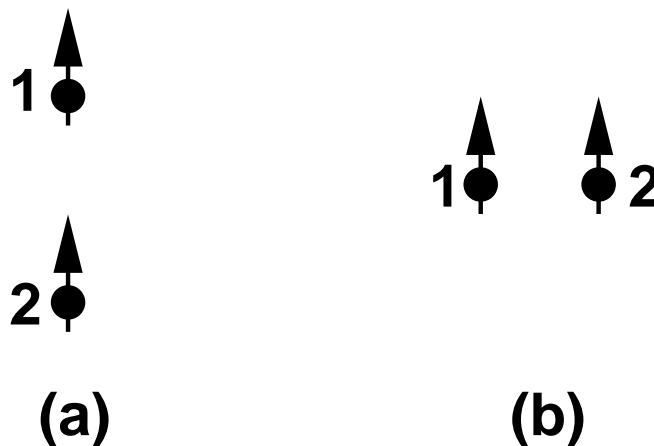
Figure 6.1: (a) $\hat{r} \| \hat{z}$ (b) $\hat{r} \perp \hat{z}$

To elucidate the effect of the tensor force on the structure of the deuteron let's consider the quadrupole moment, for which we need to use the $M = 1$ state. The dominant S-D interference term in the quadrupole moment has $M_S = 1$. So the spins of both the two nucleons are predominantly aligned parallel to $\hat{z}$. Let's simply take $\vec{\sigma}_1 = \vec{\sigma}_2 = +\hat{z}$, and then $\vec{\sigma}_1 \cdot \vec{\sigma}_2 = +1$. Then we need to consider the relative orientation of $\hat{r}$, and we will focus (see Fig. 1.1 on two extreme cases: (a) $\hat{r} \| \hat{z}$ and (b) $\hat{r} \perp \hat{z}$.

In case (a) $\vec{\sigma}_1 \cdot \hat{r} = \vec{\sigma}_2 \cdot \hat{r} = 1$, so we have $S_{12} = +2$ for this geometrical arrangement. This is a prolate configuration so we expect $Q > 0$ for case (a). In case (b) we have $\vec{\sigma}_1 \cdot \hat{r} = \vec{\sigma}_2 \cdot \hat{r} = 0$ so $S_{12} = -1$ and the oblate shape relative to the $\hat{z}$ axis would imply $Q < 0$.

Since experimentally $Q > 0$, case (a) must be energetically favored which corresponds to $V_T(r) < 0$. This then gives an attractive force when the configuration is such that $S_{12} > 0$ (case (a)) and a repulsive force when $S_{12} < 0$ (case (b)).

Given $V_C(r)$ and $V_T(r)$, this is an eigenvalue problem for k^2 with a free parameter to be determined: the ratio $\frac{b}{a}$. It was shown by Rarita and Schwinger that large class of potentials can solve these equations with the constraints E_B=2.225 MeV and $Q = 0.286e$-fm^2.

6.5 Elastic Electron-Deuteron Scattering and Meson Exchange Current

Much more detailed information about the structure of the deuteron is obtained from elastic electron deuteron scattering. At high momentum transfers this reaction probes the effects of meson exchanges and special relativity, and provides some of the most important constraints on our understanding of this simple nuclear system.
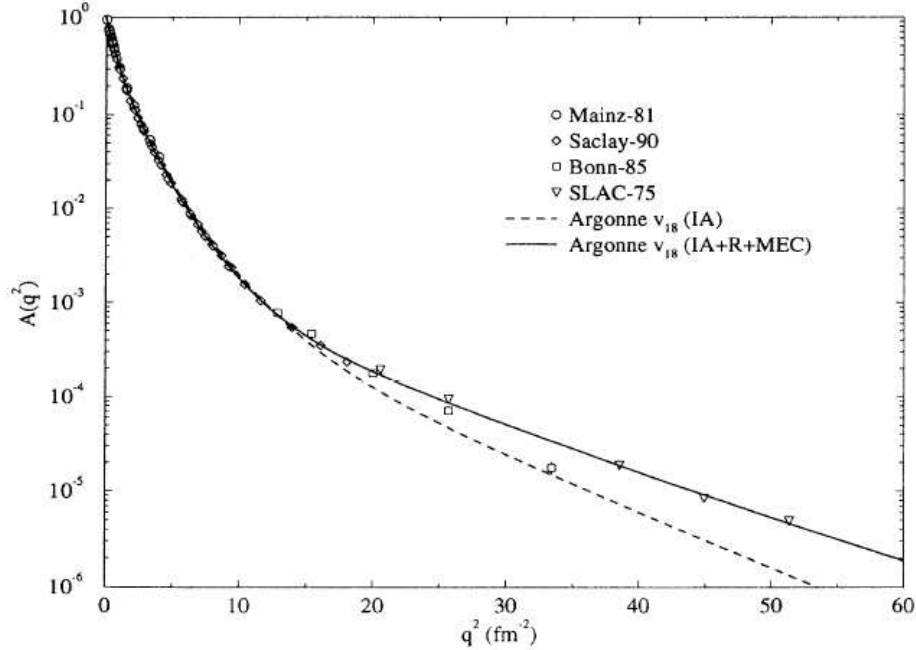
Figure 6.2: Experimental data for $A(Q^2)$ in elastic e-deuteron scattering compared with a theoretical calculations showing the sensitivity to meson exchange currents.

Deuteron has three form factors: C0 and C2, M1. In the Breit frame, we have

$$
\begin{aligned}
F_C &= \frac{1}{3\eta_1 e}\left(\langle 0|J^0|0\rangle + 2\langle +1|J^0|+1\rangle\right) \\
F_Q &= \frac{1}{2\eta\eta_1 e}\left(\langle 0|J^0|0\rangle - \langle +1|J^0|+1\rangle\right)\frac{1}{M_D^2} \\
F_M &= \frac{-1}{\sqrt{2}\eta\eta_1 e}\langle +1|J^+|0\rangle
\end{aligned}
\tag{6.61}
$$

where $J^+ = J^1 + iJ^2$, $\eta = q^2/4M_D^2$, and $\eta_1 = \sqrt{1+\eta}$. At $Q^2 = 0$, there form factors are normalized according to $F_C(0) = 1$, $F_Q(0) = Q_d$, and $F_M(0) = \mu_d M_D/m_N$.
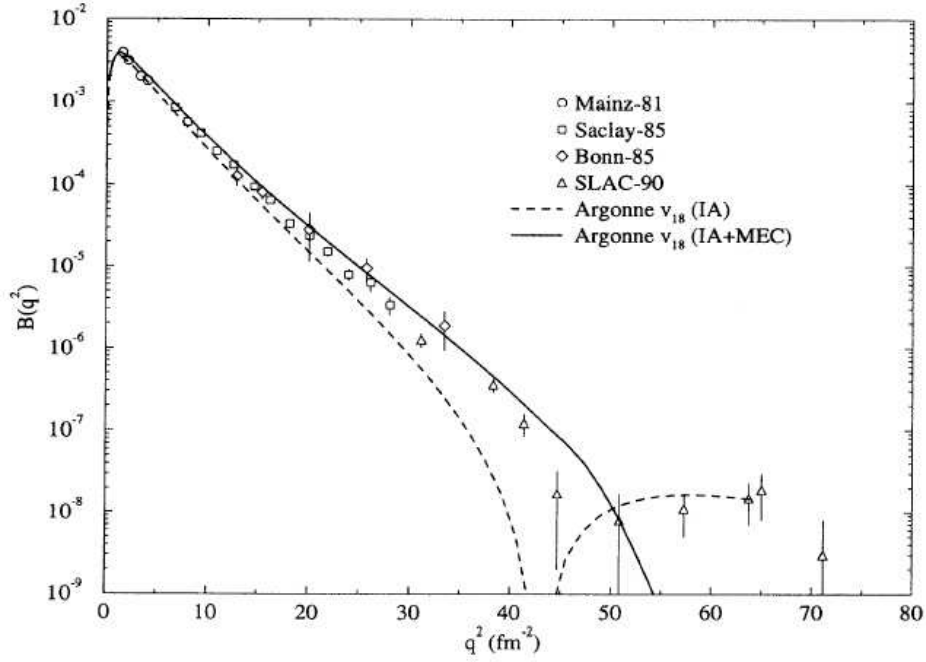
Figure 6.3: Experimental data for $B(q^2)$ from elastic e-deuteron scattering compared with theoretical calculations without (IA) and with (IA+MEC) meson exchange currents.

With elastic electron scattering on unpolarized deuterons, we can determine the following form factors,

$$A = F_C^2 + \frac{8}{9}\eta^2(F_Q M_D^2)^2 + \frac{2}{3}\eta F_M^2$$

$$B = \frac{4}{3}\eta(1+\eta)F_M^2 \qquad (6.62)$$

The scattering cross section is

$$\frac{d\sigma}{d\Omega_e} = \sigma_{\text{Mott}}\left(A(q^2) + B(q^2)\tan^2(\frac{\theta}{2})\right) \qquad (6.63)$$

Define

$$x = \frac{2\eta(F_Q^2 M_D^2)}{3F_c}$$

$$y = \frac{2\eta}{3}\left(\frac{1}{2} + (1+\eta)\tan^2(\frac{\theta}{2})\right)\left(\frac{F_M}{F_C}\right)^2 \qquad (6.64)$$

Then the tensor polarization is defined as

$$t_{20} = -\sqrt{2}\frac{x(x+2) + y/2}{1 + 2(x^2+y)} \qquad (6.65)$$

The electron-deuteron cross section depends on three elastic form factors of the deuteron: the charge form factor $G_C(Q^2)$ (monopole), the quadrupole form factor $G_Q(Q^2)$, and the magnetic form factor $G_M(Q^2)$.

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} \left\{A(q^2) + B(q^2)\tan^2\frac{\theta}{2}\right\}, \tag{6.66}$$

where

$$A(q^2) = G_C^2(q^2) + \frac{8}{9}\tau^2 G_Q^2(q^2) + \frac{2}{3}\tau G_M^2(q^2) \tag{6.67}$$

$$B(q^2) = \frac{4}{3}(1+\tau)\tau G_M^2(q^2) \tag{6.68}$$

$$\tau = \frac{-q^2}{4M_D^2} \tag{6.69}$$

$A(q^2)$ and $B(q^2)$ can be separated by a series of measurements at forward and backward scattering angles and the magnetic form factor $G_M(q^2)$ can be uniquely determined from $B(q^2)$ without using additional measurements. (The separation of the charge and the quadrupole form factors is not possible without an measurement involving polarization.) Figures 1.2 and ?? show a comparison of experimental data for $A(q^2)$ and $G_M(q^2)$ with theoretical calculations based on a phenomenological N-N interaction ("Paris potential").
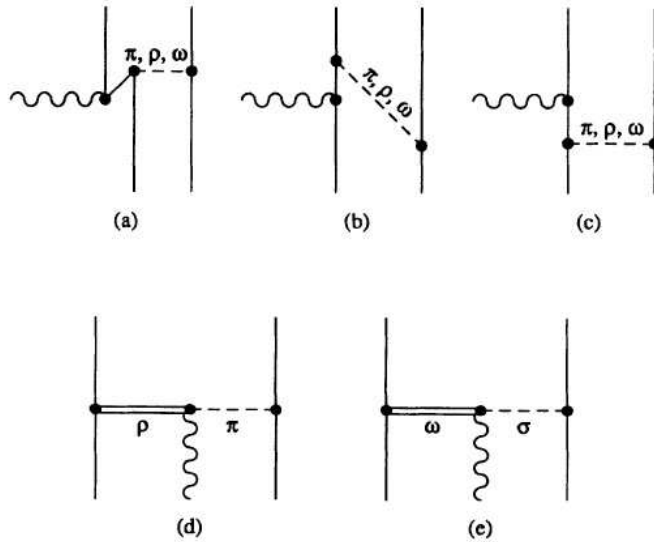

Figure 6.4: Meson exchange current diagrams required to produce the elastic form factors in e-deuteron scattering.

The graph of $A(q^2)$ in Figure 1.2shows the sensitivity to meson exchange currents. Data and calculations for $B(q^2)$ are shown in Figure 1.3. Obtaining agreement of experiment and theory in the region $q^2 > 10$ fm^{-2} requires the inclusion of meson exchange currents. While it was mentioned earlier that these effects can affect the deuteron magnetic moment at the few percent level, they become the dominant effect on $B(q^2)$ at $q^2 > 35$ fm^{-2}.

The meson exchange currents arise from processes such as those illustrated in the Feynman diagrams in Figure 1.4. Generally, one finds that at $q^2 < 30$ fm^{-2} these effects contribute mostly to isovector magnetic transitions. Of course, for elastic scattering from the deuteron only isoscalar effects contribute.

Much of the structure of the deuteron, with its rather large mean charge radius of 2 fm, is governed by the pion exchange force between the two nucleons as shown in the previous section.

6.6 Problems

1. Derive the pole of the scattering amplitude f at low energy and show that it is $E = -\hbar^2/2a^2m^2$.
2. Calculate the magnetic moment of the deuteron assuming it is either in pure S state or pure D state.
3. Derive the quadrupole moment of the deuteron in terms of the radial wave function integrals.

Chapter 7

Bulk Nuclear Properties and Nuclear Matter

The universe contains a remarkably wide variety of atomic nuclei, with mass numbers A (the sum of the numbers of proton Z and the neutron N) up to 250. While there are many interesting properties details that differentiate these nuclei from each other, there is also a powerful set of systematic trends and general properties that provide an important and useful framework for understanding the basic structure of nuclei. There properties are essentially determined by the so-called mean-field approach, in which one nucleon experiences a field which is the sum of the interaction with many other nucleons. This mean field property is a reflection that the density of the nucleons are relatively low and the interaction between the nucleons are relatively week. As such, the nucleon-nucleon correlation is small. The Hartree-Fock Mean field theory is main theoretical tool for dealing with systems with little correlations. Beyond that, the nucleon-nucleon correlations can be calculated using Bethe-Goldstone equations. These equations differ from free-space Schrodinger equation in that many-body Pauli blocking effects are taken into account. These theoretical tools are best illustrated in the example of nuclear matter in which the Coulomb potential is turned off and there are equal number of protons and neutrons. The finite size effect of the nuclear system will be taken up in the next Chapter.

7.1   Nuclear Radii and Densities

One of the first relevant properties of the nucleus was determined by Rutherford: the radius is only of order a few fm. The next major step was the use of electron scattering to accurately characterize the charge distribution of nucleons and nuclei. This technique was pioneered by Hofstadter in the 1950's. In Chapter 4, we considered the elastic scattering of electrons from nucleons. For spinless (i.e., $J = 0$) nuclei, elastic electron scattering is even simpler. The nuclear matrix element for this process is just

$$\langle 0|\hat{J}^\mu|0\rangle \ , \tag{7.1}$$

where J^μ is the electromagnetic current operator. For elastic scattering from a $J = 0$ nucleus we find

$$\langle 0|\hat{J}^\mu|0\rangle = \delta^{\mu 0}\langle 0|\hat{\rho}|0\rangle \ , \tag{7.2}$$

so that only the charge operator contributes. The electron scattering cross section is then

$$\frac{d\sigma}{d\Omega} = \sigma_{MOTT} \cdot |F(\vec{q})|^2 \; ; \quad (J = 0 \rightarrow J = 0) \tag{7.3}$$

where

$$\sigma_{MOTT} \equiv \frac{Z^2 e^4 \cos^2 \frac{\theta}{2}}{4k^2 \sin^4 \frac{\theta}{2}} f_{\text{rec}} . \tag{7.4}$$
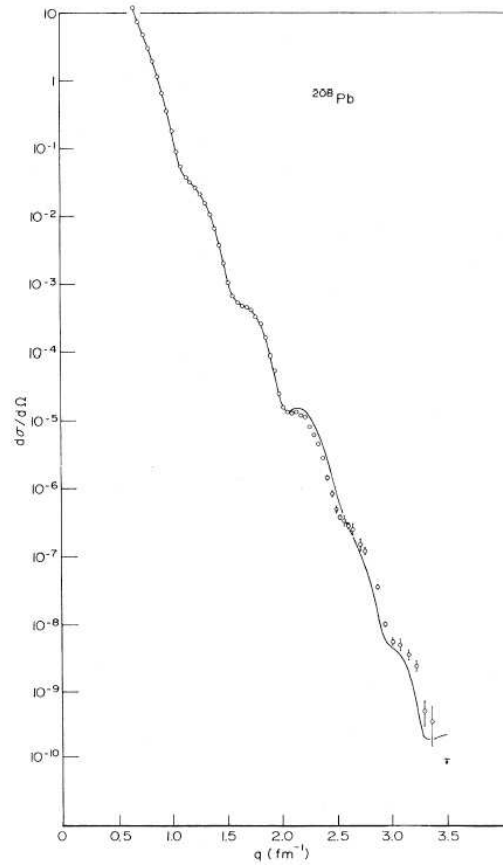
An example of this cross section is shown in Figure 2.1.

Figure 7.1: Cross section for elastic electron scattering from lead as a function of momentum transfer q along with a theoretical calculation.

The form factor $F(q)$ is directly related to the Fourier transform of the nuclear charge distribution. First define

$$\rho(q) \equiv \int \langle \rho(x) \rangle e^{-i\vec{q}\cdot\vec{x}} d^3\vec{x} , \tag{7.5}$$

where for a spin 0 nucleus $\langle \rho(x) \rangle$ is spherically symmetric. Then use the identity

$$e^{-i\vec{q}\cdot\vec{x}} = \sum_{l=0}^{\infty} (2l + 1) i^l j_l(qr) P_l(\cos\theta) \tag{7.6}$$
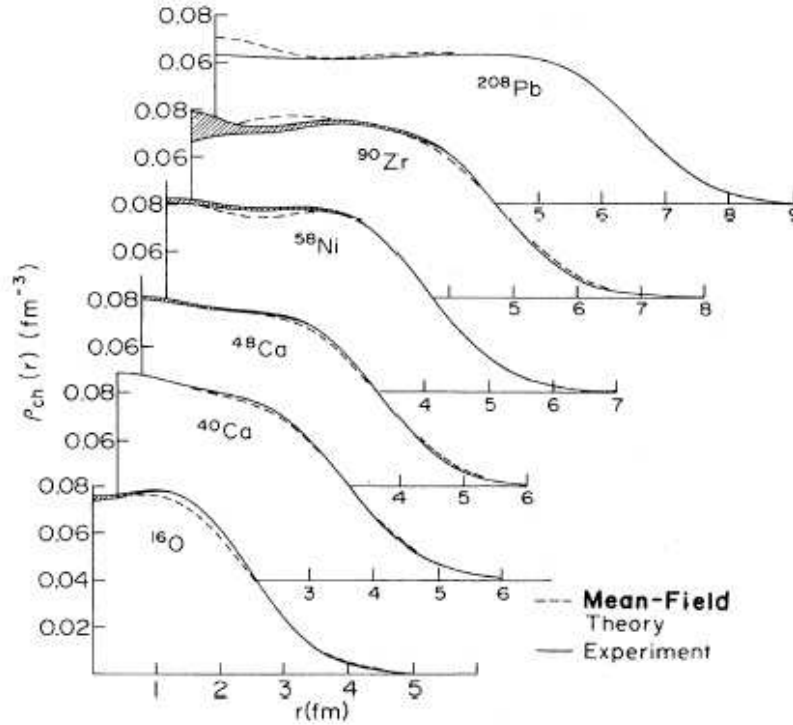
Figure 7.2: Charge density distributions for nuclei as determined from elastic electron scattering along with theoretical calculations.

and the definition

$$j_0(qr) = \frac{\sin qr}{qr} \ . \tag{7.7}$$

Only the $l = 0$ term survives due to the spherical symmetry of $\langle \rho(x) \rangle$

$$\rho(q) = \frac{4\pi}{q} \int_0^\infty \rho(r) \sin(qr) r \, dr \ . \tag{7.8}$$

We can then expand $\sin(qr)$ and obtain the relation:

$$\rho(q) = \frac{4\pi}{q} \int \rho(r)[qr - \frac{1}{6}q^3 r^3 + ...] r \, dr \tag{7.9}$$

$$= 4\pi \int \rho(r) r^2 dr - \frac{1}{6}q^2 \int r^2 \rho(r) 4\pi r^2 dr + ... \tag{7.10}$$

$$= Ze(1 - \frac{1}{6}q^2 \langle r^2 \rangle + ...) \tag{7.11}$$

We should note two important properties of $\rho(q)$:

$$\lim_{q \to 0} \rho(q) = Ze \qquad \text{(total charge)} \tag{7.12}$$

$$\lim_{q^2 \to 0} \frac{dF}{dq^2} = \frac{1}{Ze} \lim_{q^2 \to 0} \frac{d\rho}{dq^2} = -\frac{1}{6} \langle r^2 \rangle \tag{7.13}$$

or

$$\langle r^2 \rangle = -6 \frac{dF}{dq^2} \bigg|_{q^2=0} \qquad \text{(mean square charge radius)} \tag{7.14}$$

The general strategy is to measure $\rho(q)$ and Fourier transform to obtain $\rho(r)$. As shown in Figure 2.2, one finds a nuclear charge distribution that is well described by a Fermi distibution

$$\rho(r) = \frac{\rho_0}{1 + e^{(r-c)/a}} \,. \tag{7.15}$$

The basic properties of this distribution are the following.

- (a)ρ_0=constant$\sim 0.08\,e\text{fm}^{-3}$

- (b)$c = r_0 A^{\frac{1}{3}}$; $r_0 \cong 1.2\text{fm}$.

Therefore, as one adds nucleons the nuclear volume simply grows with the number of nucleons in such a way that the density of nucleons (per unit volume) is constant. That is, nuclei seem to behave like an incompressible fluid of constant density. This property is very different than that of atoms.
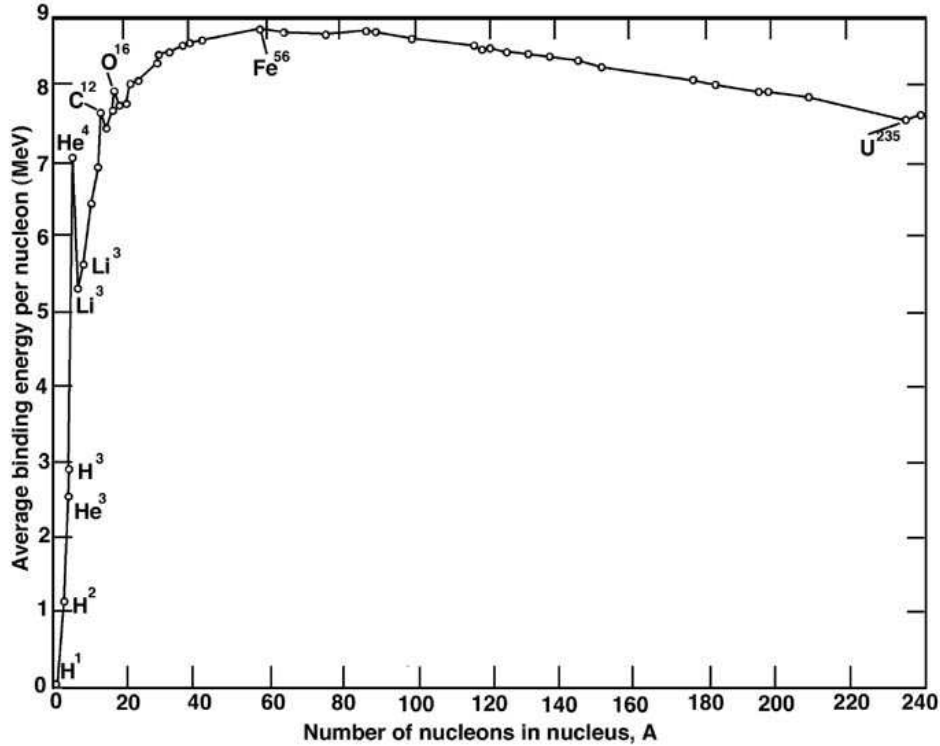

Figure 7.3: Binding energy per nucleon for stable nuclei as a function of nuclear mass number A. The binding energy saturates at the value $B/A \sim 8$ MeV/nucleon. The most stable nucleus is ^{56}Fe.