Chapter 3

Flavor Symmetry and Quark Model of Hadrons

As discussed in the previous Chapter, when the universe cools below T_C, quarks and gluons are clustered to form color-singlet hadrons. When the temperature gets sufficiently low, the anti-baryon density becomes negligible, and the excess baryon number (or baryon chemical potential) decides the density of residual baryons. Because the proton and neutrons (together called nucleons) are the lowest mass baryons, all baryon number is eventually found in the nucleons.

The fate of this excess protons and neutrons is the main subject of this book. We begin in this chapter by studying the how quarks are confined inside the nucleons, giving rise to some of the most fundamental properties, such as mass, spin and magnetic moment. In the subsequent chapter, we explore how the nucleons form stars and, together with gravity, determine the evolution of them. We would like to understand how the nuclei are formed under different conditions, and what are their fundamental properties.

The proton and neutrons are the fundamental building blocks of matter that we are most familiar with. Clearly, understanding some of their fundamental properties is critical to the understanding of the world around us. In this Chapter, we focus on the simplest approach, the quark models. Historically, the spectrum of the meson and baryons taught us a great deal about the quark substructure of hadrons. In fact, the quarks were introduced first to understand the flavor symmetry present in the states of the baryons and mesons (the so-called eight-fold way). In the end of this chapter, we will also discuss briefly the spectrum of hadrons involving at least one heavy quark. Finally, we discuss one of the puzzles in hadron spectroscopy, why we haven't seen any exotic states made of the quantum numbers of five quarks (pentaquark) or two quark plus a gluon.

3.1 Proton and Neutron, Isospin Symmetry SU(2)

The proton is the only stable hadron under the electroweak and strong interactions, and has a life time greater than 10^{33} years (the electron is the only stable lepton besides the neutrinos). It has spin-1/2, and mass 938.3 MeV/c^2. It is positively charged and has a magnetic moment 2.793 nuclear magneton μ_N $(= e\hbar/2Mc)$.

The neutron is also a spin-half particle and has a mass 939.6 MeV/c^2. The mass difference between proton and neutron is 1.293 MeV/c^2. Although it is charge-neutral, it does have a magnetic moment $-1.913\mu_N$. Free neutron is not stable; it decays weakly in 885.7 sec (about 15 min) to a proton plus an electron plus and an electron anti-neutrino (β-decay),

$$n \to p + e^- + \bar{\nu}_e \tag{3.1}$$

However, the neutrons in a stable nucleus, e.g., in the deuteron (a proton plus a neutron) or helium-3 (two protons plus a neutron), are stable. For most of the discussion, we neglect the weak interaction and consider the neutron as stable. Some of the fundamental nucleon properties are summarized in Table 3.1.

	Proton	**Neutron**
Mass	938.280 MeV	939.573 MeV
Spin	$\frac{1}{2}$	$\frac{1}{2}$
τ (Mean life)	$>10^{32}$ years	898 ± 16 sec $[n \to p + e^- + \bar{\nu}_e]$
Charge	$+e$	0
$\langle r_c^2 \rangle$	0.72 fm^2	-0.116 fm^2 (internal structure!)
Quark Structure	uud $(\frac{2}{3}\frac{2}{3}-\frac{1}{3})$	udd$(\frac{2}{3}-\frac{1}{3}-\frac{1}{3})$
μ	2.79 $\frac{e\hbar}{2M_pc}$	-1.91 $\frac{e\hbar}{2M_pc}$

Table 3.1: Basic properties of the nucleons

It is hardly an accidently that the masses of the proton and neutron are essentially the same $M_p \cong M_n$. In fact, atomic nuclei exhibit many properties supporting an interesting notion that the proton and neutron have almost identical strong interaction properties. Heisenberg was the first one who formalized this observation by introducing a symmetry, treating the neutron and proton as the two spin states of some abstract "spin"-1/2 object (isospin). Therefore the (internal) symmetry group is the same as that of the ordinary spin, SU(2).

More explicitly, the proton and neutron form an isospin double,

$$N = \begin{pmatrix} p \\ n \end{pmatrix} , \tag{3.2}$$

which transforms under isospin rotations as,

$$N' = UN , \tag{3.3}$$

where U is a 2×2 (uni-modular) unitary matrix. The generators of the SU(2) group are $\vec{I} = (I_x, I_y, I_z)$, and obey the usual SU(2) commutation relation,

$$[I_i, I_j] = i\hbar\epsilon_{ijk}I_k . \tag{3.4}$$

I^2 and I_z can be diagonalized simultaneously. Possible value of I^2 is $\hbar I(I+1)$ with $I = 0, 1/2, 1, 3/2, 2, ...$ The value of I_z for a fixed I is $-I, -I+1, ..., I$. Therefore, for the nucleon state N, $I = 1/2$; $I_z = 1/2$ for the proton and $-1/2$ for the neutron. In this representation, the generators can be taken as $\tau^a/2$ (a=1,2,3), where τ^a are the usual Pauli matrices,

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3.5}$$

As an example of an operator that depends on the isospin operators, consider the electric charge of a single nucleon,

$$\hat{Q} = \frac{1}{2}(\hat{\tau}_3 + \hat{1})e \tag{3.6}$$

so that $\hat{Q}p = p$ and $\hat{Q}n = 0$.

Other hadrons also exhibit this isospin symmetry. For example, the spin-zero π mesons come with three possible charges $+, 0$ and $-$. $\pi^\pm$ have mass 139.6 MeV, and π^0 135 MeV. They are stable if electroweak interactions are turned off. $\pi^\pm$ decays predominantly into a muon (lepton) plus a muon neutrino. And the neutral pion meson decays into 2 photons. More examples of the isospin symmetry include: spin-1/2 Σ particles in three charged states with masses 1189($+e$), 1192($0e$), 1197($-e$) MeV ($I = 1$); spin-1/2 Λ particle with mass 1116 MeV (neutral) $I = 0$; spin-1/2 Ξ particles in two charged states 1315 MeV ($-e$) and 1321 MeV ($0e$) ($I = 1/2$); spin-3/2 Δ-resonance with four different states Δ^{++}, Δ^{+}, Δ^0, and Δ^- ($I = 3/2$).

The concept of isospin symmetry in strong interactions is quite powerful, and leads to many important results in the structure and interactions of nucleons and nuclei. For the nucleons themselves, we have already noted $M_p \cong M_n$ (analogous to $E_{\frac{1}{2}} = E_{-\frac{1}{2}}$). In addition, there is substantial experimental evidence that the N-N interaction is independent of directions in "p, n" (or "isospin") space. Nuclei exhibit many properties that can be attributed to isospin symmetry. For a set of A nucleons, the total isospin is given by $\vec{T} = \frac{1}{2}\sum_{k=1}^{A} \vec{\tau}(k)$. The isospin symmetry implies that the strong interaction Hamiltonian (responsible for the N-N force) commutes with $\hat{T}$:

$$[H, \hat{T}] = 0 \ . \tag{3.7}$$

Of course we now have a more fundamental understanding of the isospin symmetry in QCD. In fact, the isospin group SU(2) is a subgroup of the larger chiral symmetry group discussed in the previous chapter, $SU_I(2) \in SU_L(2) \times SU_R(2)$. The light quarks (up and down) have very small masses compared to the intrinsic QCD scale ($m_u \cong m_d \cong$ few MeV). In addition, the gluons do not couple to the "flavor" (i.e., up-ness or down-ness) of the quarks. Therefore, the interaction between the quarks is independent of whether they are up quarks or down quarks or one of each. The combination of the flavor-blindness of QCD and $m_u \sim m_d \sim 0$ would then imply that replacing up quarks with down quarks and vice versa would not change the energy. Therefore, we would expect the proton and neutron to have the same mass (energy).

Isospin symmetry is not exact, but what breaks the isospin symmetry? The electromagnetic interaction breaks the isospin symmetry: the proton is charged but the neutron is not, and proton and neutron have different magnetic moment. The mass difference between up and down quarks also breaks the isospin symmetry. However, the isospin symmetry breaking efforts are typically very small, suppressed by $\alpha_{\rm em}/\alpha_{\rm s}$ and $(m_u - m_d)/\Lambda_{\rm QCD}$. In most cases, it is less than a percent. However, the deviation of the stability line from $Z = N$ in atomic nuclei is mostly due to isospin breaking Coulomb interaction.

3.2 Spin and Flavor Structure of Nucleons in Simple Quark Model

According to QCD, the proton is a bound state with quantum numbers of two up and one down quarks. Imagine dropping two massless up and one down quarks in color-singlet combination to the QCD vacuum, and they interact with the latter in a complicated way, settling to a proton bound state. Since in the QCD vacuum, the chiral symmetry is broken spontaneously, the QCD quarks dress themselves up with quark-antiquark pair and gluons, acquiring an effective mass of order $\Lambda_{\rm QCD}$ through the non-perturbative mechanisms. These effective quark degrees of freedom are some sort of elementary excitations of the vacuum with the QCD quark quantum numbers, and are the dubbed as *constituent quarks* in the literature. The nucleon may then be considered as a weakly interacting system of three "heavy" constituent quarks. It must be emphasized though that these arguments are heuristic at best and has not been justified from QCD directly. In particular, simplifying the problem from an infinity number of degrees of freedom to just three quarks is drastic by any measure.

Let us consider the spin and flavor structure of the proton in terms of these constituent quarks with minimal assumptions about their dynamics. In the simplest version of quark models, the constituent quarks are assumed to be non-relativistic, and move in a mean field potential, e.g., that of harmonic oscillator. One can of course introduce more complicated potentials and relativistic dynamics as well. However, it is hard to make systematic improvements without a better understanding of non-perturbative QCD. Some aspects of the simple picture, the spin and flavor part of the wave function, is less model dependent, and is justifiable in a version of QCD with a large number of color N_c (the so-called large N_c limit).

We begin by developing the SU(2) quark model of the proton and neutron. The quantum numbers of the constituent quarks are assumed to be the same as those of the QCD quarks described in Chapter 1. If one neglects the interactions between the quarks and their kinetic energy, the constituent mass must be around $M_q \sim M_p/3$, about 300 MeV. Like the electron in a hydrogen atom, the quarks have the lowest energy in the spherically symmetric s-wave state.

The color wave function must be an SU(3) singlet, therefore, if we label the color index of a single quark by i, which can take values 1, 2, 3 or red, green, blue, then the three-quark singlet state has a normalized color wave function $\frac{1}{\sqrt{6}}\epsilon^{i_1 i_2 i_3}$, where the first index refers to the color of the first quark, and so on. Clearly, this wave function is antisymmetric under the exchange of particle labels, and is normalized to 1. Since the total wave function of the quarks must be antisymmetric, the spin-flavor and spatial part of must be symmetric. In non-relativistic models, the ground states have quarks with zero orbital angular momentum, and therefore the spatial part of the wave function is symmetric. Thus Fermi statistics requires that the spin-flavor part be symmetric.

Let's begin with the maximum spin and isospin projection $S_z = \frac{3}{2}$ and $T_z = \frac{3}{2}$ state

$$|u\uparrow u\uparrow u\uparrow\rangle. \tag{3.8}$$

where the first quark is an up quark with spin up etc. This spin and flavor wave function is already symmetric in particle labels. Clearly it corresponds to $J = T = 3/2$, an excited states of the nucleon with electric charge $Q = 2e$. The so-called Δ^{++} resonance found in photo-nucleon and pi-nucleon reactions corresponds to this state. The mass of the resonance is around 1232 MeV. If we define the isospin wave function $|\psi^T_{\frac{3}{2}\frac{3}{2}}\rangle = |uuu\rangle$ and the spin wave function, $|\psi^J_{\frac{3}{2}\frac{3}{2}}\rangle = |\uparrow\uparrow\uparrow\rangle$, the above state can also be written as

$$|\psi_{\Delta^{++}}\rangle = |\psi^T_{\frac{3}{2}\frac{3}{2}}\rangle \otimes |\psi^S_{\frac{3}{2},\frac{3}{2}}\rangle \otimes |\psi_{\rm color}\rangle. \tag{3.9}$$

The other $J_z = S_z$ states are obtained by appying $\hat{S}_-$ to the above state,

$$|\psi^S_{\frac{3}{2}\frac{1}{2}}\rangle = \frac{1}{\sqrt{3}}[|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle] \tag{3.10}$$

Similarly,

$$|\psi^S_{\frac{3}{2}-\frac{1}{2}}\rangle = \frac{1}{\sqrt{3}}[|\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\rangle] \tag{3.11}$$

$$|\psi^S_{\frac{3}{2}-\frac{3}{2}}\rangle = |\downarrow\downarrow\downarrow\rangle \tag{3.12}$$

Therefore,

$$|\psi_{\Delta^Q}\rangle = |\psi^T_{\frac{3}{2}T_z=Q-\frac{1}{2}}\rangle \otimes |\psi^S_{\frac{3}{2}S_z}\rangle \times |\psi_{\rm color}\rangle \tag{3.13}$$

where $Q = 2, 1, 0, -1$ designates the charge state of the Δ.

The general case of combining 3 spin $\frac{1}{2}$ objects must result in $2^3 = 8$ independent states. Four of them are the $J = \frac{3}{2}$ states we considered for the Δ, which all have the property of being completely symmetric under permutations of any two quarks. We can construct 4 additional states that have total angular momentum $J = \frac{1}{2}$, but they are not eigenstates of permutation symmetry. We classify them by the partial permutation symmetry associated with the particles (i.e., quarks) labeled 1 and 2. The 8 states formed by combining 3 $S = \frac{1}{2}$ objects can be symbolized as

$$\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} \rightarrow \mathbf{4} \oplus \mathbf{2} \oplus \mathbf{2} \tag{3.14}$$

are displayed in Table 3.2.

$J_z = \frac{3}{2}$	$J_z = \frac{1}{2}$	$J_z = -\frac{1}{2}$	$J_z = -\frac{3}{2}$
$\uparrow\uparrow\uparrow$	$\frac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow)$	$\frac{1}{\sqrt{3}}(\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow + \downarrow\downarrow\uparrow)$	$\downarrow\downarrow\downarrow$
	$\psi_{\frac{1}{2}\frac{1}{2}}(\mathcal{S}) \equiv \frac{1}{\sqrt{6}}[(\uparrow\downarrow + \downarrow\uparrow)\uparrow -2\uparrow\uparrow\downarrow]$	$\psi_{\frac{1}{2}-\frac{1}{2}}(\mathcal{S}) \equiv -\frac{1}{\sqrt{6}}[(\uparrow\downarrow + \downarrow\uparrow)\downarrow -2\downarrow\downarrow\uparrow]$	
	$\psi_{\frac{1}{2}\frac{1}{2}}(\mathcal{A}) \equiv \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\uparrow$	$\psi_{\frac{1}{2}-\frac{1}{2}}(\mathcal{A}) \equiv \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\downarrow$	

Table 3.2: States formed by combination of 3 spin $\frac{1}{2}$ objects, or representations of SU(2).

The states $\psi(\mathcal{S})$ and $\psi(\mathcal{A})$ are known as "mixed symmetry" states. Although they have obvious even or odd symmetry under permutation of objects 1 and 2 (P_{12}), they are not eigenstates of permutations associated with objects 2 and 3 (P_{23}). For example,

$$P_{23}\psi_{\frac{1}{2}\frac{1}{2}}(\mathcal{A}) = \frac{1}{\sqrt{2}}(\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow) = \frac{1}{2}\left[\psi_{\frac{1}{2}\frac{1}{2}}(\mathcal{A}) - \sqrt{3}\psi_{\frac{1}{2}\frac{1}{2}}(\mathcal{S})\right] \tag{3.15}$$

which, however, is still orthogonal to $\psi_{\frac{3}{2}\frac{1}{2}}$. We now will proceed to construct the wave function of the nucleon ($T = \frac{1}{2},\ S = \frac{1}{2}$) using the mixed symmetry states $\psi(\mathcal{S})$ and $\psi(\mathcal{A})$ in Table 3.2. The spin states will be designated as $\psi^S(\mathcal{A})$ and $\psi^S(\mathcal{S})$, the corresponding isospin states (i.e., $T = 1/2$) by $\psi^T(\mathcal{A})$ and $\psi^T(\mathcal{S})$.

Consider the products $\psi^S(\mathcal{S})\psi^T(\mathcal{S})$ and $\psi^S(\mathcal{A})\psi^T(\mathcal{A})$, which are both manifestly symmetric under P_{12}.

$$\hat{P}_{23}\,\psi^S_{\frac{1}{2}\frac{1}{2}}(\mathcal{S})\psi^T_{\frac{1}{2}\frac{1}{2}}(\mathcal{S}) = \frac{1}{4}\left(\psi^S_{\frac{1}{2}\frac{1}{2}}(\mathcal{S}) + \sqrt{3}\psi^S_{\frac{1}{2}\frac{1}{2}}(\mathcal{A})\right)\left(\psi^T_{\frac{1}{2}\frac{1}{2}}(\mathcal{S}) + \sqrt{3}\psi^T_{\frac{1}{2}\frac{1}{2}}(\mathcal{A})\right) \tag{3.16}$$

$$\hat{P}_{23}\,\psi^S_{\frac{1}{2}\frac{1}{2}}(\mathcal{A})\psi^T_{\frac{1}{2}\frac{1}{2}}(\mathcal{A}) = \frac{1}{4}\left(\psi^S_{\frac{1}{2}\frac{1}{2}}(\mathcal{A}) - \sqrt{3}\psi^S_{\frac{1}{2}\frac{1}{2}}(\mathcal{S})\right)\left(\psi^T_{\frac{1}{2}\frac{1}{2}}(\mathcal{A}) - \sqrt{3}\psi^T_{\frac{1}{2}\frac{1}{2}}(\mathcal{S})\right) \tag{3.17}$$

The following combination is symmetric under P_{23},

$$\hat{P}_{23}\left[\frac{1}{\sqrt{2}}\left(\psi^S_{\frac{1}{2}\frac{1}{2}}(\mathcal{S})\psi^T_{\frac{1}{2}\frac{1}{2}}(\mathcal{S}) + \psi^S_{\frac{1}{2}\frac{1}{2}}(\mathcal{A})\psi^T_{\frac{1}{2}\frac{1}{2}}(\mathcal{A})\right)\right] = \frac{1}{\sqrt{2}}\left(\psi^S_{\frac{1}{2}\frac{1}{2}}(\mathcal{S})\psi^T_{\frac{1}{2}\frac{1}{2}}(\mathcal{S}) + \psi^S_{\frac{1}{2}\frac{1}{2}}(\mathcal{A})\psi^T_{\frac{1}{2}\frac{1}{2}}(\mathcal{A})\right)\ , \tag{3.18}$$

which is therefore a totally symmetric state with $S = T = 1/2$.

We can now write the proton wave function as follows

$$|\psi^p_{\frac{1}{2}\frac{1}{2}}\rangle = \frac{1}{\sqrt{2}}\left(\psi^S_{\frac{1}{2}\frac{1}{2}}(\mathcal{S})\psi^T_{\frac{1}{2}\frac{1}{2}}(\mathcal{S}) + |\psi^S_{\frac{1}{2}\frac{1}{2}}(\mathcal{A})\psi^T_{\frac{1}{2}\frac{1}{2}}(\mathcal{A})\right) \times |\psi_{\rm color}\rangle\ . \tag{3.19}$$

This is a state of good $S = \frac{1}{2}, T = \frac{1}{2}$, and is total antisymmetric under particle interchanges. It is straightforward to verify that this recipe works also for $S_z = -\frac{1}{2}$ and $T_z = -\frac{1}{2}$. If we multiply out the spin and flavor part of the wave function, we have

$$\begin{aligned}
|p\uparrow\rangle &= \sqrt{\frac{1}{18}}\,[|u\uparrow u\downarrow d\uparrow\rangle + |u\downarrow u\uparrow d\uparrow\rangle - 2|u\uparrow u\uparrow d\downarrow\rangle \\
&\quad +|u\uparrow d\uparrow u\downarrow\rangle + |u\downarrow d\uparrow u\uparrow\rangle - 2|u\uparrow d\downarrow u\uparrow\rangle \\
&\quad +|d\uparrow u\uparrow u\downarrow\rangle + |d\uparrow u\downarrow u\uparrow\rangle - 2|d\downarrow u\uparrow u\uparrow\rangle] \times |\psi_{\rm color}\rangle \\
&= \frac{1}{\sqrt{3}}\,[|u\uparrow u\downarrow d\uparrow\rangle_S - 2\,|u\uparrow u\uparrow d\downarrow\rangle_S] \times |\psi_{\rm color}\rangle\ ,
\end{aligned} \tag{3.20}$$

which is explicit in the quark spin and flavor structure.

The quark wave function of mesons can be obtained as follows: For SU(2), the iso-singlet $I = 0$ combination is $\bar{q}\mathbf{1}q \sim (\bar{u}u + \bar{d}d)/\sqrt{2}$, where $\bar{q} = (\bar{u}, \bar{d})$. The isotriplet $I = 1$ is $\phi^i \sim \bar{q}\tau^i q$, with $i = 1, 2, 3$, where τ^i are 3 Pauli matrices,

$$\begin{aligned}
\phi^1 &= (\bar{u}d + \bar{d}u)/\sqrt{2}\ , \qquad (3.21)\\
\phi^2 &= i(\bar{d}u - \bar{u}d)/\sqrt{2}\ , \\
\phi^3 &= (\bar{u}u - \bar{d}d)\sqrt{2}\ .
\end{aligned}$$

And the states with definite charges have definite isospin projection $\phi^0 = \phi^3$

$$\begin{aligned}
\phi^+ &= \frac{1}{\sqrt{2}}(\phi^1 - i\phi^2) = \bar{d}u\ , \\
\phi^- &= \frac{1}{\sqrt{2}}(\phi^1 + i\phi^2) = \bar{u}d\ .
\end{aligned} \tag{3.22}$$

3.3 Strange Quark and SU(3) Flavor Symmetry

It was discovered in early 60's that some strong interaction particles can be characterized by a strangeness quantum number, conserved in strong and electromagnetic interactions. For example, consider the spin-1/2 baryons: The nucleon isospin doublet has strangeness $S = 0$, the isospin singlet Λ and triplet Σ have strangeness $S = -1$, and the isospin doublet Ξ has strangeness $S = -2$. The masses difference between the single-strange particles and the nucleon is

$$M_\Lambda - M_N = 177 \text{ MeV}, \quad M_\Sigma - M_N = 254 \text{ MeV} . \tag{3.23}$$

Similarly, the doubly strange Ξ is heavier than the Λ

$$M_\Xi - M_\Lambda = 203 \text{ MeV} . \tag{3.24}$$

These mass differences ~ 200 MeV are much smaller than the mass themselves. Therefore if there were no small perturbation which splits the masses, the eight spin-1/2 baryons would have been degenerate.

The strangeness represents a new quantum number describing the state of particles. It is related to the electric charge, baryon number B, and isospin I_3 through,

$$Q = \frac{1}{2}(B + S) + I_3 , \tag{3.25}$$

where, the combination $B + S$ is called the hypercharge, denoted by Y. One can arrange the 8 spin-1/2 baryons in terms of the third component of the isospin and the hypercharge in a hexagon pattern shown in Figure 3.4.

There are also eight-fold degeneracies in meson spectrum. An isospin doublet K^0 and K^+ has strangeness +1, the isospin singlet η and triplet π have strangeness 0, and the isospin doublet K^- and $\bar{K}^0$ has strangeness -1. Here mass difference is much greater than the mass themselves. In the next chapter, we explain this from the so-called chiral perturbation theory. Other eight-fold degeneracies involving for example, the vector mesons ($J = 1$): ρ ($I = 1$), K^* ($I = 1/2$), ω ($I = 0$).

The $J = 3/2$ baryons show similar degeneracy behavior with

$$M_{\Sigma(1385)} - M_\Delta = 153 \text{ MeV} \tag{3.26}$$

$$M_{\Xi(1530)} - M_{\Sigma(1385)} = 145 \text{ MeV} \tag{3.27}$$

$$M_{\Omega^-} - M_{\Xi(1530)} = 142 \text{ MeV} \tag{3.28}$$

And there are ten of them forming a decuplet. These states are displayed in Figure 3.4. The triple-strange Ω^- particle with $S = -3$ is a famous prediction that was subsequently discovered in 1963.

All these multiplets suggest an internal SU(3) symmetry of hadron spectrum, which includes SU(2) isospin symmetry as a subgroup. To describe these SU(3) multiplets, the quark model of the nucleon can be generalized to the flavor SU(3) group by adding the *strange quark* with strangeness $S = -1$. The baryons with $S = -1$ contain one strange quark and these with $S = -2$ contain two. In the limit that the strange quark is massless ($m_s \to 0$), the flavor-blindness of the gluon interactions of QCD would result in perfect SU(3) symmetry. The states corresponding to the $J = 1/2$ baryon octet would be degenerate (neglecting coulomb forces), and similarly for

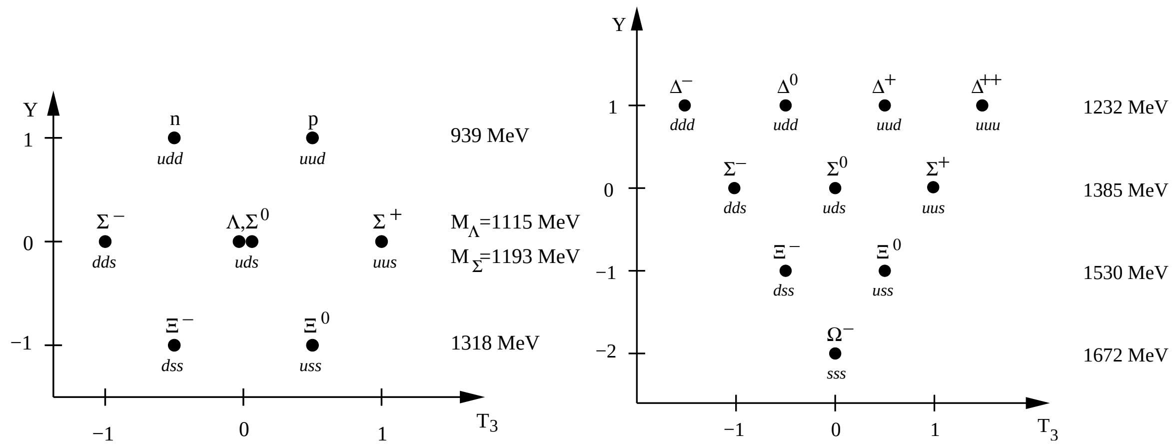

Figure 3.1: States of the $J = 1/2$ baryon octet are displayed in hypercharge (Y) vs. third isospin component (T_3)

the $J = 3/2$ decuplet. This symmetry is broken by the finite mass of the strange quark. The result is that these multiplets exhibit excellent isospin symmetry (since $m_u \simeq m_d \simeq 0$), but states containing strange quarks have masses increased by amounts that are integral multiples of m_s. Although the strange quark is heavier (by about 150 MeV) than the up and down quarks, this difference is much smaller than Λ_{QCD}, and so we still observe an approximate SU(3) symmetry in the baryon spectrum.

In the SU(3) quark model, the three flavors of quarks form a fundamental representation of SU(3). Let's see how we can construct representations of higher dimensions. Consider a second order tensor $q^i q^j$, where indices i and j each transform like a 3. It can be decomposed into symmetric and antisymmetric parts. The anti-symmetric part has 3 independent component, and the symmetric part has **6**. The **3** component one transforms like $\bar{3}$. Higher-dimensional representations can be obtained by constructing higher-order tensors. One can symmetrize and anti-symmetrize the indices of these tensors. Different types of symmetrizations yield different irreducible representations. For example, a third order tensor, or three quarks, $q^i q^j q^k$ has four different type of symmetrization,

$$\mathbf{3} \times \mathbf{3} \times \mathbf{3} = (\bar{\mathbf{3}} + \mathbf{6}) \times \mathbf{3} = \mathbf{1} + \mathbf{8} + \mathbf{8} + \mathbf{10} \tag{3.29}$$

where **1** is totally antisymmetric, **8**'s have mixed symmetry of different types, and **10** is totally symmetric. Here **8** and **10** explain the eight and ten-fold cluster of baryons.

Permutation symmetry of the resulting combinations will need to be addressed by taking appropriate products with the spin combinations (still SU(2)). There are 8 mixed symmetry states that are symmetric under P_{12}, 8 mixed symmetry states that are antisymmetric under P_{12}, and a singlet that is completely antisymmetric. The baryon octet wave function is obtained by multiplying the mixed symmetric spin states by the corresponding mixed flavor states, analogous to the treatment of SU(2) above.

The **10** represents the symmetric combinations such as uuu, ddd, sss and 7 others. These symmetric combinations are paired with the symmetric $J = \frac{3}{2}$ to form completely symmetric spin-flavor states. This forms the wave functions for the $J = \frac{3}{2}$ decuplet, including the Δ discussed above, along with Σ(1385MeV), Ξ(1530MeV), and Ω^-.

Considering the construction of meson wave functions. It is clear that $q^i\bar{q}^i$ transform like a singlet of SU(3), i.e. invariant under the flavor rotation. The product $q^i\bar{q}^j$, after subtracting the trace, transforms like as an **8** dimensional irreducible representation. Thus one has

$$\mathbf{3}\times\bar{\mathbf{3}}=\mathbf{1}+\mathbf{8}\ . \tag{3.30}$$

In fact, the eight generators of SU(3) form this 8-dimensional adjoint representation. The 8 here explains the 8-fold clusters of mesons. The singlet is $\bar{q}q \sim (u\bar{u}+d\bar{d}+s\bar{s})/\sqrt{3}$, which will be called η_0. The octet can be constructed as $\phi^i \sim \bar{q}\lambda^i q$. One has, for example, K^+ as $u\bar{s}$, K^- as $s\bar{u}$, K^0 as $d\bar{s}$, $\bar{K}^0$ as $s\bar{d}$. A flavor-neutral octet combination is

$$|8,1\rangle=(u\bar{u}+d\bar{d}-2s\bar{s})/\sqrt{6}\ . \tag{3.31}$$

which is called η_8. In reality, SU(3) symmetry is not eact, and there is a mixing. Physical η and η' are mixtures of these two states η_0 and η_8. Finally, $\phi^i\lambda^i$ has the following matrix representation,

$$\phi^i\eta^i == \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\pi^0+\frac{\eta^0}{\sqrt{3}}\right) & \pi^+ & K^+ \\ \pi^- & \frac{1}{\sqrt{2}}\left(-\pi^0+\frac{\eta^0}{\sqrt{3}}\right) & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta^0 \end{pmatrix}\ . \tag{3.32}$$

3.4 Magnetic Moments and Electromagnetic Transitions

Using the spin-flavor wave functions, we calculate a number of observables for the nucleon and other hadrons. Let us first consider the simplest quantity, the charges of the proton and neutron.

$$\begin{aligned} Q_p &= \langle p|\sum_{i=1}^{3} e_i|p\rangle \\ &= 3\langle p|e_1|p\rangle \\ &= 3\times\frac{1}{18}[e_u(1+1+4)\times 2+e_d(1+1+4)]=2e_u+e_d \end{aligned} \tag{3.33}$$

where in the second line we have used the fact that all particle labels are equivalent. In the third line, we have observed that the every term in the wave function is an eigenstate of the charge operator, and considered the contribution of all individual terms in the wave function. The final result is 1 for the proton. For the neutron, one just switches e_u and e_d, which results in zero.

Magnetic moments and other properties of the baryons can be calculated in a similar way. The magnetic moment operator of up and down quarks is given by

$$\begin{aligned} \hat{\mu}_z &= \sum_i \left[\left(\hat{t}_3(i)+\frac{1}{2}\right)\mu_u+\left(\frac{1}{2}-\hat{t}_3(i)\right)\mu_d\right]\hat{\sigma}_z(i) \\ &= (\mu_u-\mu_d)\left[\sum_i \hat{t}_3(i)\hat{\sigma}_z(i)\right]+\frac{1}{2}(\mu_u+\mu_d)\left[\sum_i\hat{\sigma}_z(i)\right] \end{aligned} \tag{3.34}$$

in which μ_u and μ_d are the magnetic moments of the up and down quarks, respectively. The matrix element of this operator is then

$$\langle\hat{\mu}_z\rangle=3(\mu_u-\mu_d)\langle\hat{t}_3(3)\hat{\sigma}_z(3)\rangle+\frac{1}{2}(\mu_u+\mu_d)\ . \tag{3.35}$$

Baryon	Magnetic Moment	quark-model expression	fit
p	2.793 ± 0.000	$\frac{4}{3}\mu_u - \frac{1}{3}\mu_d$	input
n	-1.913 ± 0.000	$\frac{4}{3}\mu_d - \frac{1}{3}\mu_u$	input
Λ	-0.613 ± 0.004	μ_s	input
Σ^+	2.458 ± 0.010	$\frac{4}{3}\mu_u - \frac{1}{3}\mu_s$	2.67
Σ^-	-1.160 ± 0.025	$\frac{4}{3}\mu_d - \frac{1}{3}\mu_s$	-1.09
Σ^0	unknown	$\frac{2}{3}(\mu_u + \mu_d) - \frac{1}{3}\mu_s$	0.79
Ξ^0	-1.250 ± 0.014	$-\frac{1}{3}\mu_u + \frac{4}{3}\mu_s$	-1.43
Ξ^-	-0.651 ± 0.003	$-\frac{1}{3}\mu_d + \frac{4}{3}\mu_s$	-0.49

Table 3.3: The magnetic moment of the octet baryons and quark model fit.

Again every term in the proton wavefunction is an eigenstate of $t_3\sigma_z$, it is then easy to see,

$$\langle\psi^p_{\frac{1}{2}\frac{1}{2}}|\hat{t}_3(3)\hat{\sigma}_z(3)|\psi^p_{\frac{1}{2}\frac{1}{2}}\rangle = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{18}\right) = \frac{5}{18} \tag{3.36}$$

Therefore,

$$\begin{aligned} \mu_p &= \langle\hat{\mu}_z\rangle = \frac{5}{6}(\mu_u - \mu_d) + \frac{1}{2}(\mu_u + \mu_d) = \frac{4}{3}\mu_u - \frac{1}{3}\mu_d \\ \mu_n &= -\frac{5}{6}(\mu_u - \mu_d) + \frac{1}{2}(\mu_u + \mu_d) = -\frac{1}{3}\mu_u + \frac{4}{3}\mu_d. \end{aligned} \tag{3.37}$$

For $\mu_q \propto Q_q$ (quark charge), we have $\mu_u = -2\mu_d$, and

$$\frac{\mu_p}{\mu_n} = -\frac{3}{2}\,. \tag{3.38}$$

Note that this is every close to the experimental value $\frac{\mu_p}{\mu_n} = -1.46$. If we further assume that the quark mass is $\frac{1}{3}$ of the nucleon mass and has the usual Dirac moment

$$\mu_i = \frac{Q_i\hbar}{2M_ic}\,;\ (i = \text{up, down})\,, \tag{3.39}$$

then $\mu_u = 2$ nuclear magnetons (n.m.) and $\mu_d = -1$ n.m.. This gives the prediction $\mu_p = 3$ n.m. and $\mu_n = -2$ n.m., which is also quite successful.

Using the quark model wave functions, we find the magnetic moments of other baryon octet members as shown in Table. If one makes the assumption that $\mu_u = -2\mu_d$, there are two parameters which can be used to fit the experimental data. The result is surprisingly good.

Other notable successes of the SU(6) wave function include calculating the decay of Δ (spin-3/2) resonance to the nucleon state. Δ is a spin-3/2 state belonging to the baryon decuplet. The spins of all quarks are aligned in one direction. It can decay electromagnetically to the nucleon,

$$\Delta \to p + \gamma\ . \tag{3.40}$$

From the spins of the nucleon and Δ, we know that the angular momentum of the photon can either be 1 or 2. Taking into account parity, this photon can be either magnetic dipole or coulomb or electric quadrupoles. If the ground state wave functions of the nucleon and delta are spatially symmetric, there can be no quadrupole deformation. Thus the transition shall go almost exclusively by the magnetic dipole, through which one of the quarks flips its spin. Experimentally, $E2/M1$ ratio is about 2%!

3.5 Neutron Beta Decay and Axial-Current Coupling Constants

As can be seen in Table 3.1, the neutron decays into a proton by the weak interaction process known as beta decay:

$$n \to p + e^- + \bar{\nu}_e \ . \tag{3.41}$$

The energy released is

$$Q = M_n - M_p - m_e - m_\nu \ . \tag{3.42}$$

We will see later that the neutrino is very light so we may take $m_\nu \simeq 0$, and thus

$$Q = M_n - M_p - m_e \ = \ (939.57 - 938.28 - 0.51)\,\mathrm{MeV} \tag{3.43}$$

$$= \ 0.78\,\mathrm{MeV}. \tag{3.44}$$

Since Q is much smaller than the mass of the proton, it is a good approximation to neglect its recoil energy. Therefore, the energy released is shared between the electron and neutrino, and the maximum kinetic energy of the e^- is $Q = 0.78$ MeV.

Beta decay proceeds via the weak interaction, and so we may use first order perturbation theory as an excellent approximation. We utilize Fermi's golden rule and write the decay width as

$$d\Gamma = \frac{2\pi}{\hbar}|H_{fi}|^2\frac{d\rho}{dE_f} \ , \tag{3.45}$$

where the density of final states may be written

$$\frac{d\rho}{dE_f} = \frac{d^3p_e}{(2\pi\hbar)^3}\cdot\frac{d^3p_\nu}{(2\pi\hbar)^3}\cdot\delta(E_0 - E_e - E_\nu) \tag{3.46}$$

where $E_0 \equiv Q + m_e =$ maximum total β energy E_e.

In the standard model, the effective weak Hamiltonian responsible for the neutron decay is

$$H_{\rm eff} = \frac{G_F}{\sqrt{2}}\cos\theta_c\bar{u}\gamma^\mu(1-\gamma_5)d\bar{e}\gamma_\mu(1-\gamma_5)\nu \ , \tag{3.47}$$

where $\bar{u}\gamma^\mu d$ and $\bar{u}\gamma^\mu\gamma_5 d$ are quark vector and axial vector currents, respectively. To show the flavor structure, one introduces SU(3) flavor currents

$$j^\mu_a = \bar{q}\gamma^\mu t_a q; \qquad j^\mu_{a5} = \bar{q}\gamma^\mu\gamma_5 t_a q \tag{3.48}$$

where t^a is the flavor SU(3) generator. Then the current involved in the above effective hamiltonian can be written as $j^\mu_{1+i2} = (j^\mu_1 + ij^\mu_2)/2$.

The matrix element of the quark currents between the proton and neutron states can be calculated using isospin symmetry. It is easy to show that

$$\langle P|\bar{u}\Gamma d|N\rangle = \langle P|(\bar{u}\Gamma u - \bar{d}\Gamma d)|P\rangle \ , \tag{3.49}$$

where Γ is a generic Dirac matrix. Therefore,

$$\langle P|\bar{u}\gamma^\mu d|N\rangle = g_V \bar{U}_P \gamma^\mu \tau^+ U_N \tag{3.50}$$

where U_P and U_N are the Dirac spinor for the proton and neutron respectively, and $g_V = 1$ from the vector current conservation. For the axial current, one defines

$$\langle P|\bar{u}\gamma^\mu \gamma_5 d|N\rangle = g_A \bar{U}_P \gamma^\mu \gamma_5 \tau^+ U_N \ , \tag{3.51}$$

where g_A is called the neutron decay constant.

If the neutron and final states are unpolarized, one can calculate the matrix element of the hamiltonian,

$$|H_{\rm eff}|^2 = G_F^2 \left(g_V^2 (1+\cos\theta) + 3g_A^2 (1 - \frac{1}{3}\cos\theta) \right) \ . \tag{3.52}$$

where θ is the angle between electron and neutrino momenta. The contribution from the vector current comes from $\mu = 0$ component and is usually called the Fermi interaction, and that from the axial comes from $\mu = i$ components, and is usually called the Gamow-Teller interaction.

Since we won't detect the e^- direction, nor the ν (direction or energy) we will integrate over the e^-, ν angles (note that $p_\nu = E_\nu$ for $m_\nu = 0$):

$$d\Gamma = \frac{1}{2\pi^3} |H_{\rm eff}|^2 (E_0 - E_e)^2 p_e^2 dp_e \ . \tag{3.53}$$

Allowing for the Coulomb effect on e^- will modify the density of final states, and this has the effect that we must multiply our result by $F(E_e) \equiv$ "Fermi Function". In the absence of the Coulomb interaction on the final e^-, $F(E_e) = 1$. Finally, we use that $p_e dp_e = E_e dE_e$ to obtain

$$d\omega = \frac{1}{2\pi^3} |H_{\rm eff}|^2 F(E_e)(E_0 - E)^2 p_e E_e dE_e \tag{3.54}$$

To compute the total decay rate, we integrate over dE_e to obtain

$$\Gamma = \frac{|H_{\rm eff}|^2 m_e^5}{2\pi^3 \hbar^7} \frac{1}{m_e^5} \int_{m_e}^{E_0} F(E) \ (E_0 - E)^2 pE dE \ . \tag{3.55}$$

We then define the Fermi Integral

$$f(E_0) \equiv \frac{1}{m_e^5} \int_{m_e}^{E_0} F(E) \ (E_0 - E)^2 pE dE \tag{3.56}$$

which are standard tabulated functions. Using the experimental mean life $\tau = 1/\Gamma = 886$ sec, one finds axial coupling constant $g_A = 1.27$.

One can calculate g_A in the constituent quark model. Taking $\mu = i$, the vector current in non-relativistic approximation becomes $\vec{A} = \psi^\dagger \vec{\sigma} \tau^+ \psi$. Using isospin symmetry, g_A can be calculated as the matrix element of $\vec{A} = \psi^\dagger \vec{\sigma} \tau_3 \psi$ in the proton state. A simple quark wave function gives,

$$\langle P|A^z|N\rangle = \langle P|\psi^\dagger \sigma^z \tau_3 \psi|P\rangle = (5/3)\bar{U}_P \sigma^z U_P \ . \tag{3.57}$$

Therefore one has $g_A = 5/3$, which is significantly larger than the experimental value.

Decay	Axial coupling	SU(3) expression	fit
$n \to p\ell\nu$	1.2664 ± 0.0065	$F + D$	1.266
$\Sigma \to \Lambda\ell\nu$	0.602 ± 0.014	$\sqrt{2/3}D$	0.602
$\Lambda \to p\ell\nu$	$-.890 \pm 0.015$	$-\sqrt{3/2}F - \sqrt{1/6}D$	-0.896
$\Sigma \to n\ell\nu$	0.341 ± 0.015	$-F + D$	0.341
$\Xi \to \Lambda\ell\nu$	0.306 ± 0.061	$\sqrt{3/2}F - \sqrt{1/6}D$	0.306
$\Xi \to \Sigma\ell\nu$	0.929 ± 0.0012	$\sqrt{1/2}(F + D)$	0.929

Table 3.4: The semi-leptonic decay of hyperons and SU(3) flavor symmetry

The other members of the baryon octet have similar semi-leptonic decay modes which can be analyzed analogously. Now the weak interaction current involves also the strange quark decay,

$$j^\mu = \cos\theta_C \bar{u}\gamma^\mu(1-\gamma_5)d + \sin\theta_C \bar{u}\gamma^\mu(1-\gamma_5)s \ , \tag{3.58}$$

which is a combination of $1+2i$ and $4+5i$ components of the flavor currents. In the approximation that the SU(3) flavor symmetry is good, one can express the matrix elements of the octet axial currents in terms of just two parameters F and D: The baryon octet can be written in a matrix form

$$B = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\Sigma^0 + \frac{\Lambda^0}{\sqrt{3}}\right) & \Sigma^+ & p \\ \Sigma^- & \frac{1}{\sqrt{2}}\left(-\Sigma^0 + \frac{\Lambda^0}{\sqrt{3}}\right) & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda^0 \end{pmatrix} . \tag{3.59}$$

When a matrix element of an octet operator O_a is taken between the baryon octets, there are two possible invariants

$$\langle B'|O_a|B\rangle = D\ \mathrm{Tr}(T_a\{B, \bar{B}'\}) + F\ \mathrm{Tr}T_a[B, \bar{B}'] \tag{3.60}$$

This is a generalization of the Wigner-Eckart theorem to the SU(3) case.

All measured semileptonic decays of baryon octet are shown in Table xx. The second column gives the axial current decay constant. The SU(3) expression for the decay constant in terms of F and D is shown in the third column. Using the experimental data, one can perform a least chi-square fit, yielding $F = 0.47 \pm 0.01$ and $D = 0.79 \pm 0.01$. In the quark model, the SU(6) spin-flavor wave function gives $F = 2/3$ and $D = 1$. The reduction from the quark model result is generally viewed as a relativistic effect.

3.6 Excited Baryons in Non-Relativistic Quark Model

So far we have focused on the ground states of baryons. Experimentally, many excited states with different spin and parity have been observed through πN and γN scattering. To describe excited baryons including the masses of the baryon decuplet, one must introduce dynamics for the constituent quarks. One of the simplest choices is the non-relativistic (NR) quark model, in which quarks are assumed to move in an harmonic oscillator potential. The choice of this potential is mainly due to its simplicity in separating out the center-of-mass motion. It is a good approximation to a more realistic potential with linear confinement and one-gluon exchange. To go beyond, one can take into account the difference by a residual two-body interaction.

The zero-th order hamiltonian is,

$$H_0 = \frac{1}{2m}(\vec{p}_1^2 + \vec{p}_2^2) + \frac{1}{2m'}\vec{p}_3^2 + \frac{1}{2}k\sum_{i<j}(\vec{r}_i - \vec{r}_j)^2 \ . \tag{3.61}$$

where 1 and 2 are assumed to have the same mass, e.g, up and down quarks, and 3 is chosen to be different, convenient for a strange quark, for example. Introduce the Jacobi coordinates and center-of-mass,

$$\begin{aligned} \vec{\rho} &= \frac{1}{\sqrt{2}}(\vec{r}_1 - \vec{r}_2) \\ \vec{\lambda} &= \frac{1}{\sqrt{6}}(\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3) \\ \vec{R}_{\rm cm} &= \frac{m(\vec{r}_1 + \vec{r}_2) + m'\vec{r}_3}{(2m + m')} \ , \end{aligned} \tag{3.62}$$

where $\vec{\rho}$ is antisymmetric and $\vec{\lambda}$ symmetric in indices 1 and 2. Defining momentum $\vec{p}_\rho = m_\rho d\rho/dt$ where $m_\rho = m$, and similarly for $\vec{p}_\lambda$ with $m_\lambda = 3mm'/(2m + m')$, then the hamiltonian can be written as,

$$H = \frac{\vec{p}_\rho^2}{2m_\rho} + \frac{3}{2}k\vec{\rho}^2 + \frac{\vec{p}_\lambda^2}{2m_\lambda} + \frac{3}{2}k\vec{\lambda}^2 + \frac{\vec{P}_{\rm cm}^2}{2M} \ . \tag{3.63}$$

The center-of-mass motion is completely separated and will be ignored henceforth. There are then two types of bayron excitations associated with spatial motions, ρ and λ.

The energy levels of the 3-quark system is simply (for $m = m'$)

$$E_N = (N + 3)\hbar\omega \ , \tag{3.64}$$

where $N = N_\rho + N_\lambda = (2n_\rho + \ell_\rho) + (2n_\lambda + \ell_\lambda)$ and $\omega = \sqrt{3k/m}$. The orbital angular momentum of a state is obtained by coupling $\vec{\ell}_\rho$ and $\vec{\ell}_\lambda$, $\vec{L} = \vec{\ell}_\rho + \vec{\ell}_\lambda$, and parity is $P = (-1)^{\ell_\rho + \ell_\lambda}$.

For $N = 0$, all quarks are in s-wave, and we obtain octet and decuplet baryons. This is the 56plet in spin-flavor SU(6) language. The spatial wave function is

$$\psi_{00}^S = \frac{\alpha^3}{\pi^{3/2}} e^{-\alpha^2(\rho^2+\lambda^2)/2} \tag{3.65}$$

where $\alpha = (3km)^{1/4}$ and the two subscripts refer to N and L, respectively. If we label the states according to $|X^{2S+1}L_\pi J^P\rangle$ with X = N or Δ, S the total spin, L total orbital angular momentum, π the permutation symmetry of the spatial wave function, and J^P the total angular momentum and parity, then we have

$$\begin{aligned} |N^2 S_S \tfrac{1}{2}^+\rangle &= |\psi_{00}^S\rangle \frac{1}{\sqrt{2}}(\phi_N^\rho \chi_{\frac{1}{2}}^\rho + \phi_N^\lambda \chi_{\frac{1}{2}}^\lambda) \ , \\ |\Delta^4 S_S \tfrac{3}{2}^+\rangle &= |\psi_{00}^S\rangle \phi_\Delta^S \chi_{\frac{3}{2}}^S \ , \end{aligned} \tag{3.66}$$

where ϕ and χ are flavor and spin wave functions, respectively.

The masses of the baryon decuplet are heavier than those of the baryon octet. Therefore, the strong force is not independent of the the spin of the quarks after all. We must add spin-spin interactions (magnetic moment-moment interactions). There are two possible mechanisms to generate the hyperfine splitting, one-gluon exchange interaction and one pion exchange. Both seem to be able to described the physics phenomenologically. For example, the one-gluon exchange hyperfine interaction goes like,

$$V^{ij} = \frac{2\alpha_s}{3m_i m_j}\left[\frac{8\pi}{3}\vec{S}_i\cdot\vec{S}_j\delta^3(r_{ij}) + \frac{1}{r_{ij}^2}S_{ij}^{(2)}\right] , \tag{3.67}$$

where the second term represents the tensor interaction. Using the first-order perturbation theory, one has

$$M_\Delta - M_N = 2\sqrt{2}\alpha_s\alpha_0^3/3m^2\sqrt{\pi} . \tag{3.68}$$

Taking $\omega = 0.5$, $m = 0.33$ GeV, one needs $\alpha_s = 0.9$ to fit the difference.

For $N = 1$, the angular momentum is $L = 1$, we have negative parity P-wave states. The spatial wave function has mixed symmetry, for example,

$$\psi_{lm_\rho}^{M_\rho} = \sqrt{\frac{\alpha_\rho^5\alpha_\lambda^3}{\pi^3}}\rho_{m_\rho}e^{-(\alpha_\rho^2\rho^2+\alpha_\lambda^2\lambda^2)/2} \tag{3.69}$$

In this case, the spin-flavor wave function also need be mixed symmetric. This is possible either for the spin part to be mixed symmetric or the isospin part to be mixed symmetric or both to be mixed symmetric. One then finds the following states,

$$\begin{aligned}
|N^4P_M(\tfrac{1}{2}^-,\tfrac{3}{2}^-,\tfrac{5}{2}^-)\rangle &= \chi^S_{\frac{3}{2}}\frac{1}{\sqrt{2}}\left(\phi_N^\rho\psi_1^{M_\rho} + \phi_N^\lambda\psi_1^{M_\lambda}\right)\\
|N^2P_M(\tfrac{1}{2}^-,\tfrac{3}{2}^-)\rangle &= \frac{1}{2}\left[\phi_N^\rho\left(\psi_1^{M_\rho}\chi^\lambda_{\frac{1}{2}} + \psi_1^{M_\lambda}\chi^\rho_{\frac{1}{2}}\right) + \phi_N^\lambda\left(\psi_1^{M_\rho}\chi^\rho_{\frac{1}{2}} - \psi_1^{M_\lambda}\chi^\lambda_{\frac{1}{2}}\right)\right]\\
|\Delta^2P_M(\tfrac{1}{2}^-,\tfrac{3}{2}^-)\rangle &= \phi^S_\Delta\frac{1}{\sqrt{2}}\left(\psi_1^{M_\rho}\chi^\rho_{\frac{1}{2}} + \psi_1^{M_\lambda}\chi^\lambda_{\frac{1}{2}}\right)
\end{aligned} \tag{3.70}$$

These states form a part of the spin-flavor **70** of SU(6). The average energy of the states is at 1600 MeV. Including appropriate hyperfine splitting, one finds a $N^*1/2^-$ and $N^*3/2^-$ states are at 1500 MeV range, consistent with experimental data.

For $N = 2$, we have even-parity excited states. Consider first total $L = 0$. In this case, either one of the radial quantum numbers is equal to 1, corresponding to orbitally excited states, or one has $\ell_\rho = 1$ and $\ell_\lambda = 1$, and they couple to $L = 0$. One can have $L = 1$, when these two angular momentum couple to 1. There are three possibilities to get $L = 2$, either one of the ℓ_ρ and ℓ_λ is 2, or two $\ell = 1$ couples to 2. One can generate many states when combined with spin and flavor wave functions appropriately. These states can be used as a basis to diagonalize the residual and hyperfine interactions. One of the calculation is shown in Fig. xx, where comparison is made with the resonances collected in Particle Data Group.

Some remarks about the comparison between data and the NR quark model. First, there are more theoretical states than what experimentally has been observed. Two possibilities exist: either the other resonances are too broad and too weak to be seen, or the models with 3-quarks do not represent appropriately the correct number of degrees of freedom in spectroscopy. In fact, there are suggestions in the literature that perhaps a quark-diquark system represents better the low-energy excitations of the baryons.

Second remark is that the quark model states cannot be compared with resonance masses and width directly. In fact, resonances are part of the scattering states. One has to couple the quark

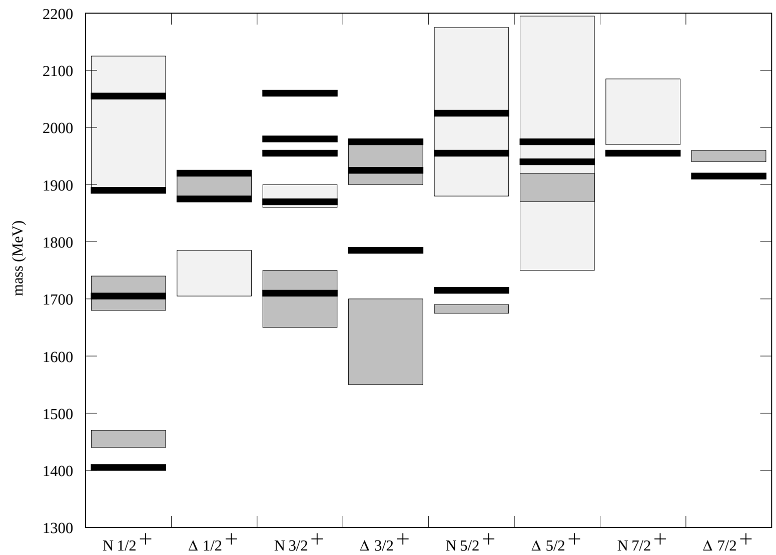

Figure 3.2: The energy-levels of baryon resonances (shown as bars) in simple harmonic oscillator potential mixed by residual and hyperfine interactions. The PDG resonances are shown by shaded boxes with 3 and 4 star states shaded darkar than 1 and 2 star states.

model states with the continuum states to generate proper resonances in reality. Therefore, an appropriate comparison with experimental data is to use the quark model states to generate the S-matrix elements of scattering.

3.7 The MIT Bag Model

Despite its simplicity and successes, the NR quark model has a number of undesirable features. First of all, the kinetic energy of the quark $p^2/2m \sim 1/2mr^2$ is roughly the same order as the constituent mass and is hardly non-relativistic. Second, although the quark confinement is in some sense reflected in the harmonic oscillator potential, this was motivated mostly by the simplicity of math, not by the physics of the QCD vacuum. The so-called MIT bag model proposed by K. Johnson et al provides a fresh picture on the quark motion in hadrons.

As we have discussed earlier, the QCD vacuum sees to behave like a dual superconductor, i.e., it repels color electric fields, such that color dielectric constant vanishes $\epsilon_c = 0$ (dual Meissner effect). At the same time, it is a perfect magnet with $\mu_c = \infty$. As such when quarks are inserted in the vacuum, the latter will likely create cavity to let the quarks move freely inside and the color electric field is confined. Such a cavity is behave very much like a perturbative vacuum. However, it costs energy to create a cavity like this, and the bag constant B is introduced to described the energy of the cavity per unit volume.

The quarks move freely in the bag to a good approximation. Therefore, we can solve the free Dirac equation (neglecting the small current quark masses)

$$i\gamma^\mu \partial_\mu \psi = 0 \ . \tag{3.71}$$

To confine the quark in the bag, one uses that condition that the scalar potential is infinite outside of the bag. [One can also try to use the fourth component of a vector potential, however, it can be shown that this cannot confine the quark entirely this way.] This analogous to the example of 3-dimensional infinity potential well in NR quantum mechanics. There is, however, a difference: the wave function does not vanish at the boundary. What vanishes there is the quark scalar density $\bar{\psi}\psi = 0$.

The wave function in spherical coordinate system is

$$\psi = \begin{pmatrix} g_\kappa(r)\mathcal{Y}^m_{jl}(\hat{r}) \\ if_\kappa(r)\mathcal{Y}^m_{jl'}(\hat{r}) \end{pmatrix} \tag{3.72}$$

where $\mathcal{Y}^m_{jl} = \sum\langle lm_l 1/2 m_s|jm\rangle Y_{lm_l}\chi_{m_s}$ is the spinor spherical harmonics, χ_{m_s} is the eigenstate of the spin-1/2 particle. Since the particle is free, the solution has the following simple form

$$\begin{aligned} g_\kappa(r) &= Nj_l(Er); \\ f_\kappa(r) &= N\frac{\kappa}{|\kappa|}j_{l'}(Er) \ , \end{aligned} \tag{3.73}$$

where $l = \kappa$ and $l' = k - 1$ for $\kappa > 0$, and $l = -(\kappa + 1)$ and $l' = -\kappa$ for $\kappa < 0$. The boundary condition leads to the following eigen-equation,

$$j_l(E_{n\kappa}R) = (-1)^\lambda j_{l+1}(E_{n\kappa}R) \ , \tag{3.74}$$

with $\lambda = 1$ for $l = \kappa = j+1/2$, and $\lambda = 0$ for $l = -(\kappa+1) = j-1/2$. κ is quantum number takes value ± 1, ± 2, ... Write $E_{n\kappa} = \omega_{n\kappa}R$, the lowest few eigenvalues correspond to $\omega = 2.04$ $(j = 1/2, l = 0)$, 3.81 $(j = 1/2, l = 1)$ etc.

The motion of the quarks in the bag generates a pressure on the surface of the bag. The pressure can be calculated as the change of energy when the volume changes

$$P = -\frac{1}{2}\frac{d(\overline{\psi}\psi)}{dr} \ . \tag{3.75}$$

This pressure is positive-definite and is a function of bag radius, larger the radius, smaller the pressure. If this is the only pressure, the bag will expand indefinitely. The bag constant B also generates a negative pressure $-B$. The bag radius is determined by the balance of these two pressures.

Even when the quark is massless, its motion in the bag will generate a magnetic moment. One can calculate this using the electric current $\vec{j} = e_q\overline{\psi}\vec{\gamma}\psi$. The result is

$$\mu = \frac{e}{2E}\left(1 - \frac{2}{3}\int_0^\infty f(r)^2 r^2 dr\right) = \frac{e_q(4x-3)}{12x(x-1)}R \tag{3.76}$$

where the first result is for any quark state, and the second is true only for the ground state. If one uses this to fit the proton's magnetic moment, one finds that $R = 1.18$fm. On the other hand, the mean square radius of the particle is

$$\langle r^2\rangle = \int_0^R (g^2 + f^2)r^4 dr \tag{3.77}$$

which gives $0.73R$ in the ground state.

The neutron decay constant can also be calculated in the bag,

$$g_A = \frac{5}{3}\int r^2 dr (f^2 - \frac{1}{3}g^2) \tag{3.78}$$

Clearly this is reduced due to the lower component of the wave function (relativistic effect).

The color fields shall satisfy the following boundary conditions

$$\hat{n} \times B = 0; \quad \hat{n} \cdot E = 0 \tag{3.79}$$

The color electric and color magnetic fields in the ground state can be calculated using perturbation theory. Using the result, we can calculate the mass splitting between nucleon and Δ resonance.

3.8 Hadrons With Heavy Quarks

So far, we have basically ignored the heavy quarks: the charm, bottom and top. However, physics involving heavy flavors is extremely rich, and is related to much of the fore-frontier research in high-energy physics in recent years. Here we make a very basic introduction to the subject. From the QCD point of view, because the mass scale of the heavy quarks is much larger than $\Lambda_{\rm QCD}$, the heavy quark systems actually present a simplification.

3.8.1 Mesons with Hidden Heavy Flavor

The first and most famous example of mesons with hidden heavy flavor is J/ψ, first discovered in 1974, which is a vector meson made of a charm and anti-charm quark. The discovery of the Υ particle in 1978(?) ushers in another series of mesons with a pair of bottom and anti-bottom quarks. The $t\bar{t}$ mesons do not have enough time to get formed because the fast top-quark (weak) decay.

The mesons formed by quark and antiquark pairs are called oniums just like the bound states of electron and positron (positronium). When masses are heavy, their average separation is short and interaction between quark and antiquark pair is dominated by one gluon exchange. Therefore, the physics of a heavy quarkonium is not too different from that of a positronium: it is a non-relativistic Coulomb bound system. In reality, when the mass of the quark is not so heavy, such as the charm quark, one has to take into account the long range confinement interaction as well (which is also important for the excited states). The simplest phenomenological potential between heavy quarks is

$$V(r) = -\frac{4}{3}\frac{\alpha_s}{r} + br \tag{3.80}$$

where b is called the string tension, which can be calculated in lattice gauge theory.

Just like the positronium case, the quantum numbers for the quarkonium states are $P = (-1)^{L+1}$, and $C = (-1)^{L+S}$. Therefore, we have the following list of states,

$$0^{++}, 0^{-+}, 1^{--}, 1^{+-}, 1^{++}, 2^{++} \tag{3.81}$$

The S-wave states inlcude 1^{--} and 0^{-+} which have been observed for both bottonium and charmonium, inluding the radial excitations. The lowest P-wave states have also been seen. More states

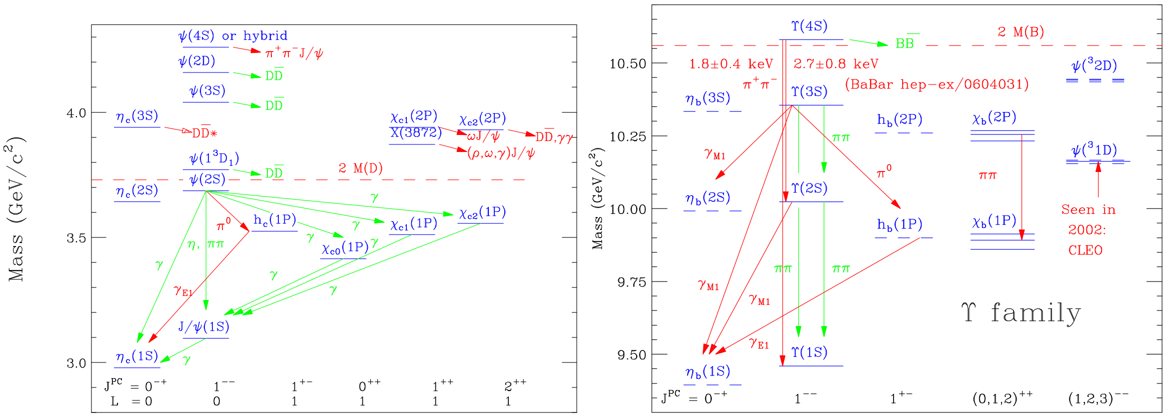

Figure 3.3: The energy-levels of baryon resonances (shown as bars) in simple harmonic oscillator potential mixed by residual and hyperfine interactions. The PDG resonances are shown by shaded boxes with 3 and 4 star states shaded darkar than 1 and 2 star states.

above the $D\bar{D}$ and $B\bar{B}$ thresholds, mesons made of a charm or bottom quark plus a light quark, have been observed. Some of the states, such as $\psi(3S)$ and $\Upsilon(4S)$ states are very useful to study $D\bar{D}$ and $B\bar{B}$ mixing. A host of new charmed states have been observed in recent years, and their interpretation in terms of quark and gluon constituents has been a bit controversial at the moment.

The vector mesons such J/ψ and Υ are quite special. They decay through the annihilations of the heavy quark pairs. This annihilation process is strongly suppressed because of the QCD interactions. Therefore, this leads to the long life-time of these hidden-flavor mesons and hence easy identification in high-energy processes.

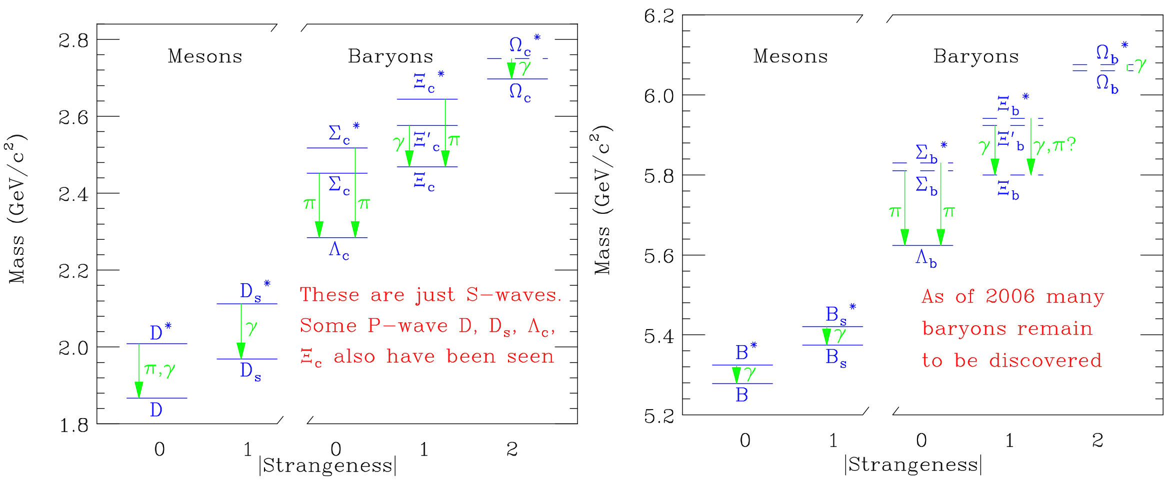

Figure 3.4: The energy-levels of baryon resonances (shown as bars) in simple harmonic oscillator potential mixed by residual and hyperfine interactions. The PDG resonances are shown by shaded boxes with 3 and 4 star states shaded darkar than 1 and 2 star states.

3.8.2 Hadrons with A Single Heavy Quark

Mesons or baryons with a single-heavy quark contain flavor quantum numbers. As such, in their ground states, they cannot decay strongly, rather through weak interactions only. In fact, weak decays of the heavy-flavored hadron have lead to much better understanding of weak interactions. Because these mesons contain light quarks, they can only be discussed heuristically in simple quarks models. However, they have an important symmetry, the so-called heavy quark symmetry: the physics of these mesons are independent of the spin and flavor of the heavy quark to the leading order approximation.

Combining a single charm quark and a light antiquark, we obtain D mesons. The low-lying D-meson states are shown in Fig. xx. The ground states have spin-parity 0^{-+}. The first excited states have quantum number 1^{-+} and are labeled with a $*$. The charm quark mostly decays to a strange quark (Cabbibo allowed), with a smaller probability decaying to a down quark. This Cabbibo suppression can be seen clearly through $D^0 \to \pi^+\pi^-$ and $D^0 \to K^-\pi^+$, with the former rate being only about 4-5 percent of the latter.

Combining a single bottom quark or antiquark with a light quark companion, we have B-mesons. The lowest B-meson states are shown in Fig. xx. The lowest B-mesons have been used extensively to study the weak interaction properties of the bottom quark. Indeed, the so-called B-factory, such as Babar and Belle, were built to create $B\bar{B}$ mesons and study their subsequent decays. The dominant decay modes of B-meson are through $b \to c$. However, there are also substantial decays through $b \to u$. B mesons can also decay leptonically, yielding the decay constants f_B, which can be directly calculated in lattice QCD. B meson decays have taught us a great deal about CKM flavor structure and CP violations in standard model.

A meson contain two-heavy quarks, B_c, has been observed recently, and has a mass 6275 MeV.

Heavy baryons can be made of one or more heavy quarks. The most famous examples are Λ_c and Λ_b mesons. However, other type of heavy baryon have been observed as well. The orbital excitations of Λ_c, Σ_c, and Ξ_c have also been observed recently. Λ_b baryon has a lift time about 1.4 ps.

3.9 Exotic Hadrons?

All hadronic states established so far have the following simple feature: mesons are made of the quantum numbers of a quark-antiquark pair, and baryons made of those of three quarks. However, QCD does not forbid states of more exotic combinations, such as the quantum numbers of two quarks plus a gluon or that of five quarks $uudd\bar{s}$. Since they is no strong evidence of their existence in experiments, they are called exotic states or exotic hadrons.

Let us first consider a type of exotic mesons. For a quark-antiquark pair, the possible quantum numbers J^{PC} are $P = (-1)^{L+1}$ and $C = (-1)^{L+S}$, where L and S are the orbital and spin quantum number. Specifically, one has

$$\begin{aligned} &0^{-+}, 0^{++}, \\ &1^{--}, 1^{+-}, 1^{++}, \\ &2^{--}, 2^{-+}, 2^{++} \\ &... \end{aligned} \tag{3.82}$$

Therefore, it is not possible to form 0^{--}, $0^{+-}, 1^{-+}, 2^{+-}, ...$ states. If one find a state with these quantum numbers, the minimal particle content would be a pair of quarks and a gluon. Therefor, these exotic states are a good indicate of gluon excitations.

The pure gluon excitations are called glue-balls. However, in QCD, glue-balls mix strongly with quark-antiquark states and hence cannot exist by themselves. Hadrons with strong glue-ball components have been suggested and looked for in experiments. There are no firm conclusion yet.

The so-called pentaquarks, hadrons made of a minimal five quark component, have generated considerable interest in recent years. Consider the process, $\gamma + n \to X + K^-$. If X is resonance, it cannot be made of just three quarks, because it at least contains an anti-strange quark. The minimal possible quark content is $uudd\bar{s}$. If it exists, it is a new type of baryon resonance. Since a stable neutron target does not exist, one can look for pentaquarks through, for example, $\gamma + d \to K^- + P + X$. A peak in the invariant mass of X is an indication of pentaquark.

Consider a pentaquark baryon $qqqq\bar{q}$. Since the color wave function has to be a singlet, $qqqq$ must be a 3. There is only one way to do this: 3 quarks couple to a singlet and the remainder quark is a 3. In this case, the spin and flavor wave function of the four quarks must be the (3,1) structure of SU(6) with 210 dimension. Decomposing 210 into $SU(3) \times SU(2)$

$$210 = [3, 0 + 1] + [\bar{6}, 1] + [15, 0 + 1 + 2] + [15, 1] \tag{3.83}$$

When coupled with $[\bar{3}, 1/2]$, the flavor representations 1, 8, 10, $\bar{1}0$, 27, 35 appear. The tensor decomposition works, for example, like this,

$$[\bar{6}, 1] \times [\bar{3}, 1/2] = [\bar{1}0 + 8, 1/2 + 3/2] \tag{3.84}$$

$\bar{1}0$ contains a particle called Θ^+ with $S = 1$, which can be identified with the state $uudd\bar{s}$. So does the 27 and perhaps even the 35.

3.10 Problem Set

1. Construct the quark model wave function for the Λ (isospin 0) and calculate its magnetic moment.
2. Calculate the weak hadronic matrix element squared for the neutron decay.
3. Calculate the SU(3) expression for the axial decay constant of $\Xi \to \Lambda \ell \nu$ in terms of F and D.
4. Calculate the quark model prediction for F and D.
5. Calculate the neutron decay constant g_A in the MIT bag model.
6. Calculate the magnetic moment in the MIT bag model.