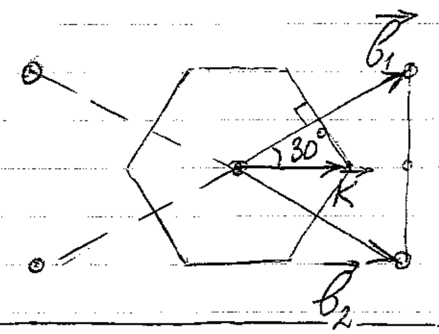
1). From $\vec{a}_1 = a_0\sqrt{3}\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\vec{a}_2 = a_0\sqrt{3}\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
we can find $\vec{b}_2 = \frac{\pi}{3a_0}(\sqrt{3}, 1)$, $\vec{b}_1 = \frac{\pi}{3a_0}(\sqrt{3}, -1)$

For vector $\vec{K}$ and analogous vectors:

$$|K| \cdot \cos 30^\circ = \frac{b_1}{2}$$

$$|K| = \frac{b_1}{2\cos 30^\circ} = \frac{4\pi}{3a_0\sqrt{3}}$$



For such a vector:

$$\varepsilon(k_x, k_y) = \pm \gamma_1 \sqrt{1 + 4\cos\left(\frac{\sqrt{3}a_0}{2} \cdot \frac{4\pi}{3a_0\sqrt{3}}\right) + 4\cos^2\left(\frac{\sqrt{3}a_0}{2} \frac{4\pi}{3a_0\sqrt{3}}\right)} =$$

$$= \pm\gamma_1 \sqrt{1 + 4\cdot\left(-\frac{1}{2}\right) + 1} = 0$$

2). By definition $\varphi_k^{\pm} = (b_1, b_2) = \frac{1}{\sqrt{2}}\left(\pm e^{i\beta/2}, e^{-i\beta/2}\right)$

$$|\langle \varphi_k^+ | \varphi_k^+ \rangle|^2 = \frac{1}{2}\left|\left(e^{-i\beta/2}, e^{+i\beta/2}\right)\begin{pmatrix} e^{i\beta/2} \\ e^{-i\beta/2}\end{pmatrix}\right|^2 = \frac{1}{2}\cdot 2 = 1$$

$$|\langle \varphi_k^- | \varphi_k^- \rangle|^2 = \frac{1}{2}\left|\left(-e^{-i\beta/2}, e^{i\beta/2}\right)\begin{bmatrix} -e^{-i\beta/2} \\ e^{i\beta/2}\end{bmatrix}\right|^2 = \frac{1}{2}\cdot 2 = 1$$

$$|\varphi_{-k}^+\rangle = \frac{1}{\sqrt{2}}\left(e^{i\beta/2 + i\pi/2}, e^{-i\beta/2 - i\pi/2}\right) = \frac{1}{\sqrt{2}}\left(ie^{i\beta/2}, -ie^{-i\beta/2}\right)$$

$$|\varphi_{-k}^-\rangle = \frac{1}{\sqrt{2}}\left(-e^{+i\frac{\beta+\pi}{2}}, e^{-i\frac{\beta+\pi}{2}}\right) = \frac{1}{\sqrt{2}}\left(-ie^{i\beta/2}, -ie^{-i\beta/2}\right)$$

$$|\langle \varphi_k^+ | \varphi_{-k}^+ \rangle|^2 = \frac{1}{2}\left|\left(e^{-i\beta/2}, e^{i\beta/2}\right)\begin{bmatrix} ie^{i\beta/2} \\ -ie^{-i\beta/2} \end{bmatrix}\right|^2 = \frac{1}{2}\cdot 0 = 0$$

$$|\langle \varphi_k^- | \varphi_{-k}^- \rangle|^2 = \frac{1}{2}\left|\left(-e^{-i\beta/2}, e^{i\beta/2}\right)\begin{bmatrix} -ie^{i\beta/2} \\ -ie^{-i\beta/2} \end{bmatrix}\right| = \frac{1}{2}\cdot 0 = 0$$

$$\langle \varphi_k^+ | \varphi_{-k}^- \rangle = \frac{1}{2}\left(e^{-i\beta/2}, e^{i\beta/2}\right)\begin{bmatrix} -ie^{i\beta/2} \\ -ie^{-i\beta/2} \end{bmatrix} = -i \Rightarrow |\langle \varphi_k^+ | \varphi_{-k}^- \rangle|^2 = \underline{1}$$

For different k_1 and k_2 we have:

$$\langle \varphi_{k_1}^p | \varphi_{k_2}^q \rangle = \left(\operatorname{sgn} p\, e^{-i\beta_1/2}, e^{i\beta_1/2}\right)\begin{bmatrix} \operatorname{sgn} q\, e^{i\beta_2/2} \\ e^{-i\beta_2/2} \end{bmatrix}\frac{1}{2} =$$

$$= \frac{1}{2}\operatorname{sgn} p \cdot \operatorname{sgn} q\, e^{i(\beta_2-\beta_1)/2} + e^{i\frac{\beta_1-\beta_2}{2}}\frac{1}{2} = \cos\frac{\beta_1-\beta_2}{2}\cdot(\operatorname{sgn} p\cdot\operatorname{sgn} q + 1)\frac{1}{2} +$$
$$+ \frac{i}{2}\sin\frac{\beta_1-\beta_2}{2}(1 - \operatorname{sgn} p \operatorname{sgn} q) \Rightarrow$$

$$\Rightarrow |\langle \varphi_{k_1}^p | \varphi_{k_2}^q \rangle|^2 = \cos^2\frac{\Delta\beta}{2}\cdot\delta_{pq} + (1-\delta_{pq})\sin^2\frac{\Delta\beta}{2},$$ where

we used $\delta_{pq} = \frac{1}{2}(1+\operatorname{sgn} p \operatorname{sgn} q)$, $1-\delta_{pq} = \frac{1}{2}(1-\operatorname{sgn} p \operatorname{sgn} q)$.

$\operatorname{sgn} p = +1$ for $p = +$ and $\operatorname{sgn} p = -1$ for $p = -$.

3) General wrapping vector has form:

$$\vec{W} = n\vec{a}_1 + m\vec{a}_2$$

For Bloch wavefunction we have

$$\Psi_k(\vec{x}+\vec{R}) = \Psi_k(\vec{x}) e^{i\vec{k}\vec{R}}$$

So $\quad k_\perp \cdot |W| + \vec{K}\cdot\vec{W} = 2\pi p \Leftrightarrow$

$$\Leftrightarrow \quad k_\perp \cdot |W| + 2\pi/3\,(n-m) = 2\pi p \quad \left(\vec{K} = \frac{4\pi}{3\sqrt{3}a_0}\hat{e}_x\right)$$

And finally $\quad k_\perp = \dfrac{2\pi}{|W|}\cdot\left(p + \dfrac{-n+m}{3}\right) = \dfrac{2\pi}{\pi d}\left(p + \dfrac{m-n}{3}\right).$

Since near $\vec{K}$ points spectrum has the linear form, we have:

$$\mathcal{E}(\vec{\varkappa}) = \pm\hbar v_F \cdot |\vec{\varkappa}| = \pm\hbar v_F\sqrt{\left(\frac{2}{d}\left(p+\frac{m-n}{3}\right)\right)^2 + k_\parallel^2} =$$

$$= \pm\hbar v_F \cdot 2/d\,\sqrt{\left(p+\frac{m-n}{3}\right)^2 + \left(\frac{k_\parallel d}{2}\right)^2}$$

We see, that for $\frac{m-n}{3}$ = integer there is no gap ($\mathcal{E}(\vec{\varkappa}) = 0$ for certain wavevectors) $\Rightarrow$ NT is metallic.

But when $\frac{m-n}{3} \neq$ integer, then there is a gap in energy spectrum of order $\mathcal{E}_{gap} \sim \frac{\hbar v_F}{d}$.